

Popper #5

← Bubble correctly!

① What is the center of the circle?

$$x^2 + (y-2)^2 = 3 \Rightarrow (x-0)^2 + (y-2)^2 = 3$$

$\downarrow_0 \quad \quad \downarrow_2$

- A. (0, 2) B. (2, 0) C. (0, 0) D. ($\sqrt{3}$, 0)

② What is the focus of parabola?

horizontal $\rightarrow y^2 = 4x - 8 = 4(x-2) \Rightarrow \begin{matrix} 4p=4 \\ p=1 \end{matrix} (1, 0) \xrightarrow[\text{2 unit right}]{\text{shifted}} (3, 0)$

- A. (1, 0) B. (3, 0) C. (1, 2) D. (0, 1)

③ What is the orientation of ellipse:

horizontal $\rightarrow \frac{x^2}{36} + \frac{y^2}{4} = 1$

A. Vertical

B. Horizontal

④ Given the ellipse $\frac{(x-2)^2}{4} + \frac{(y-1)^2}{9} = 1$, find the center. (2, 1)

A. (2, 1)

B. (4, 4)

C. (2, 3)

D. none of them.

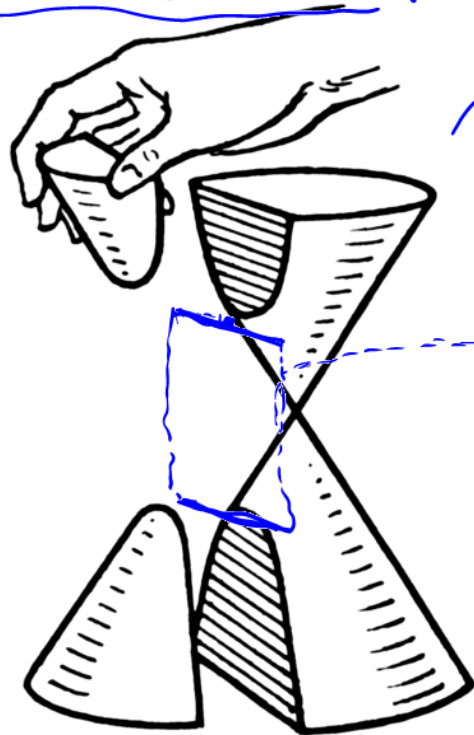
Equation for Hyperbolas: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Section 8.3 Hyperbolas

Definition: A hyperbola is the set of all points, the difference of whose distances from two fixed points is constant. Each fixed point is called a focus (plural = foci).

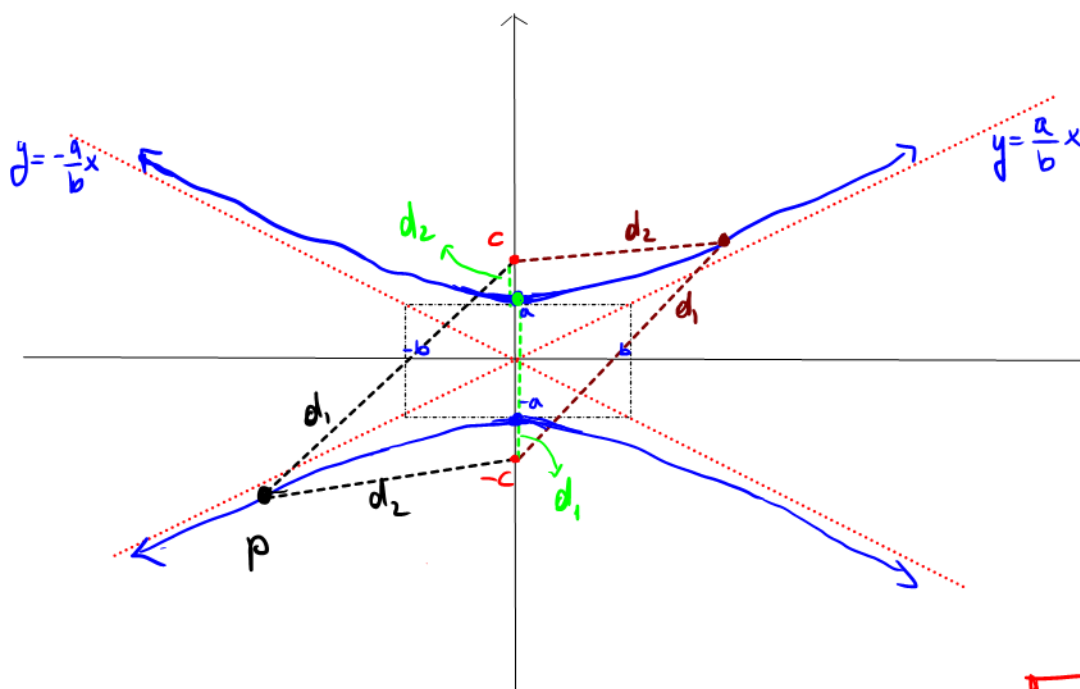
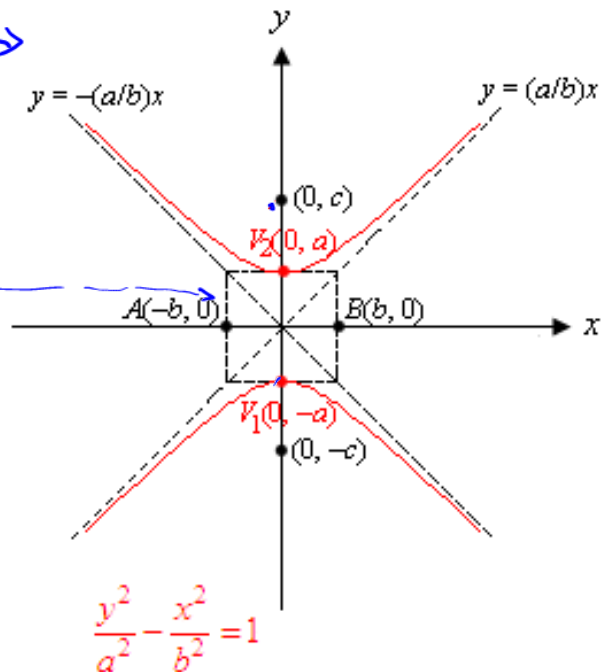
The focal axis is the line passing through the foci.

Visualize how it is built!



graph

Cross-section



By definition
 $d_1 - d_2 = \text{fixed}$
?

$d_1 - d_2 = \text{fixed}$

$d_1 = c + a$

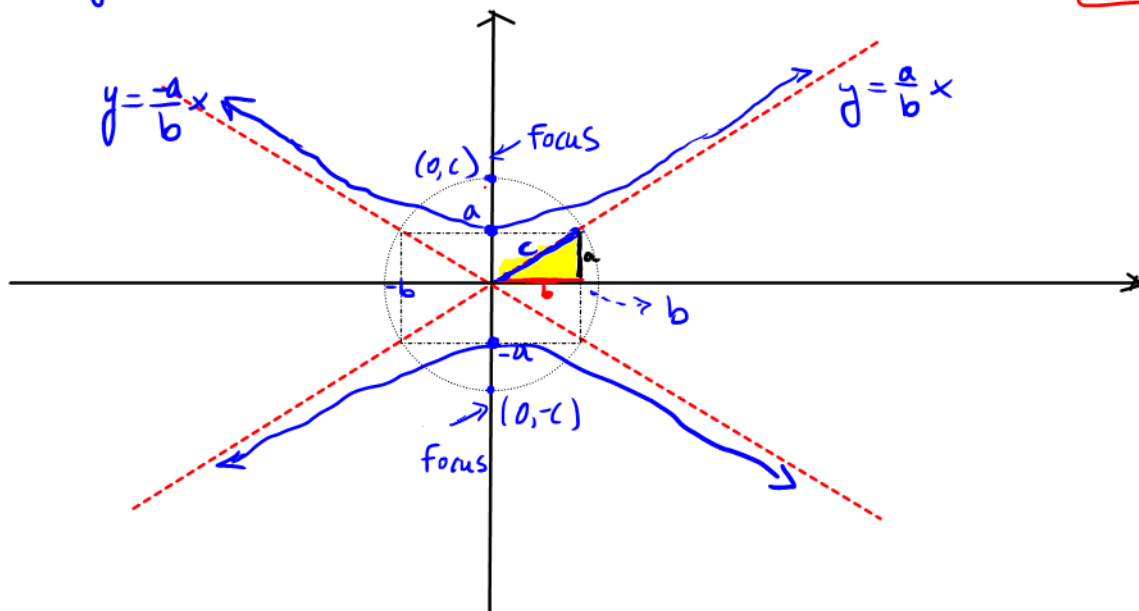
$d_2 = c - a$

$d_1 - d_2 = 2a$

$\Rightarrow d_1 - d_2 = 2a$

$\Rightarrow$ By using similar strategy as in parabolas or ellipses, we get $c^2 = a^2 + b^2$

The beauty of hyperbolas is that the foci and the corners of the rectangle are in a circle. This occurs because $c^2 = a^2 + b^2$



Ellipses

vs.

Hyperbolas

- by definition $d_1 + d_2 = 2a$
- Foci Equation $c^2 = a^2 - b^2$
(foci are between vertices)
- Vertices
 - horizontal $(a, 0), (-a, 0)$
 - vertical $(0, a), (0, -a)$
- Major axis $= 2a$
(connects vertices)
- Minor axis $= 2b$

- by definition $|d_1 - d_2| = 2a$
- Foci Equation $c^2 = a^2 + b^2$
(foci are beyond vertices)
- Vertices
 - horizontal $(a, 0), (-a, 0)$
 - vertical $(0, a), (0, -a)$
- Transverse axis $= 2a$
(connects vertices)
- Conjugate Axis $= 2b$

Ellipses

Vs

Hyperbolas (continued)

- Horizontal Ellipse

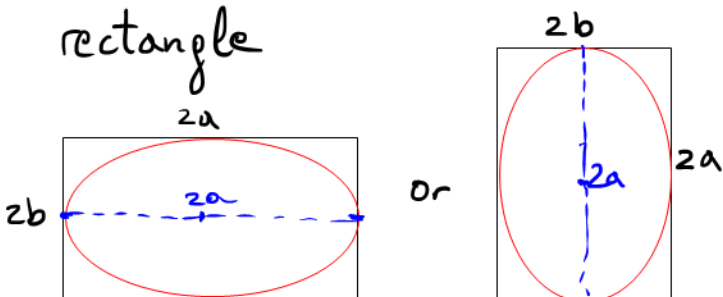
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- Vertical Ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

i.e. $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$

- Ellipse is inside the rectangle



- Horizontal Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

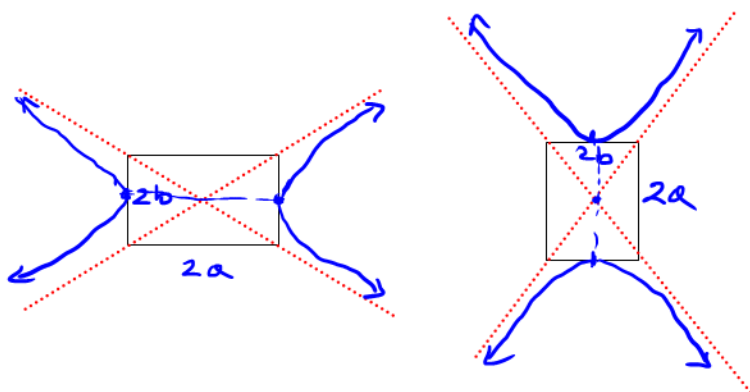
- Vertical Hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

- Slant Asymptotes

$$\underbrace{y = \pm \frac{b}{a} x}_{\text{horizontal}} \quad \text{or} \quad \underbrace{y = \pm \frac{a}{b} x}_{\text{vertical}}$$

- Hyperbola is beyond the rectangle



Standard form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Basic "vertical" hyperbola: → begins with $\frac{y^2}{a^2}$

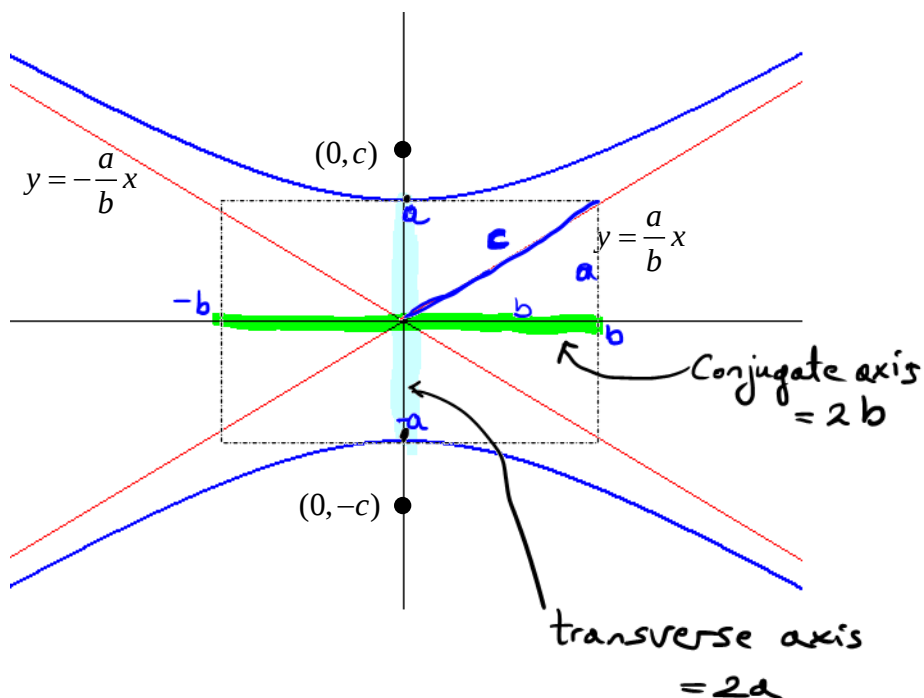
Equation: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Asymptotes: $y = \pm \frac{a}{b}x$

Foci: $(0, \pm c)$, where $c^2 = a^2 + b^2$ always

Vertices: $(0, \pm a)$

Eccentricity: $\frac{c}{a} (> 1)$



Basic "horizontal" hyperbola: → begins with $\frac{x^2}{a^2}$

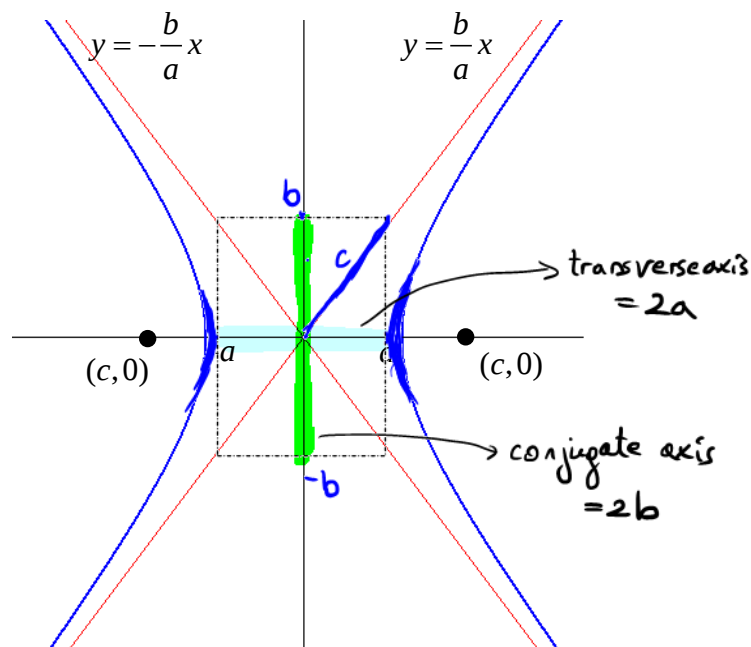
Equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Asymptotes: $y = \pm \frac{b}{a}x$

Foci: $(\pm c, 0)$, where $c^2 = a^2 + b^2$ always

Vertices: $(\pm a, 0)$

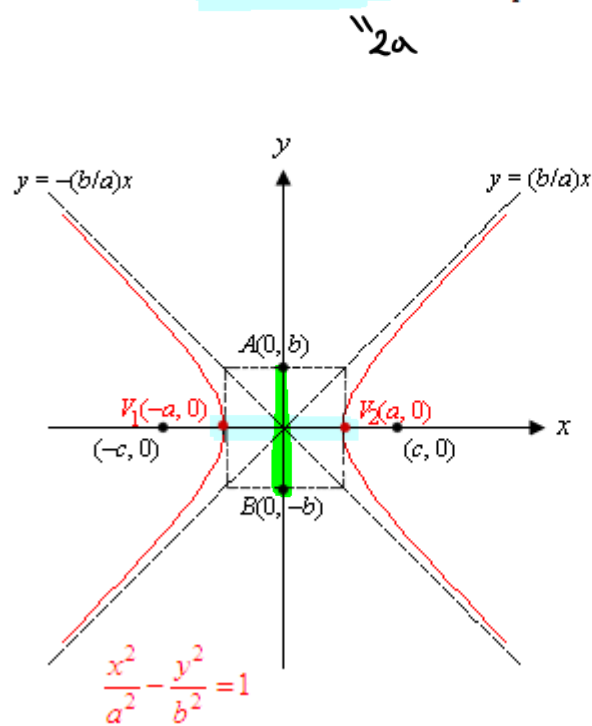
Eccentricity: $\frac{c}{a} (> 1)$



Note: The **transverse axis** is the line segment joining the two vertices. The **conjugate axis** is the line segment perpendicular to the transverse axis, passing through the center and extending a distance b on either side of the center. (These terms will make more sense after we do the graphing examples.)

Details about conjugate and transverse axis.

The conjugate axis of the hyperbola is the line segment through the center of the hyperbola and perpendicular to the transverse axis with endpoints $(0, -b)$ and $(0, b)$.



Center: $(0, 0)$

Foci: $(-c, 0)$ and $(c, 0)$, where $c^2 = a^2 + b^2$

Vertices: $V_1(-a, 0)$ and $V_2(a, 0)$

Transverse Axis: $\overline{V_1V_2}$ Length of Transverse Axis: $2a$

Conjugate Axis: $\overline{AB}$ Length of Conjugate Axis: $2b$

The eccentricity of a hyperbola is given by the formula $e = \frac{c}{a}$.

The lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are slant asymptotes for the hyperbola since it can

be shown that as $|x|$ becomes large, $y \rightarrow \pm \frac{b}{a}x$.

Graphing hyperbolas: *Let's demonstrate by doing example 1.*

To graph a hyperbola with center at the origin:

Just know how to begin!!!

- Rearrange into the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.
- Decide if it's a "horizontal" or "vertical" hyperbola.
 - if x^2 comes first, it's horizontal (vertices are on x-axis).
 - If y^2 comes first, it's vertical (vertices are on y-axis).
- Use the square root of the number under x^2 to determine how far to measure in x-direction.
- Use the square root of the number under y^2 to determine how far to measure in y-direction.
- Draw a box with these measurements.
- Draw diagonals through the box. These are the asymptotes. Use the dimensions of the box to determine the slope and write the equations of the asymptotes.
- Put the vertices at the edge of the box on the correct axis. Then draw a hyperbola, making sure it approaches the asymptotes smoothly.
- $c^2 = a^2 + b^2$ where a^2 and b^2 are the denominators.
- The foci are located c units from the center, on the same axis as the vertices.

When graphing hyperbolas, you will need to find the orientation, center, values for a , b and c , lengths of transverse and conjugate axes, vertices, foci, equations of the asymptotes, and eccentricity.

→ let's begin:

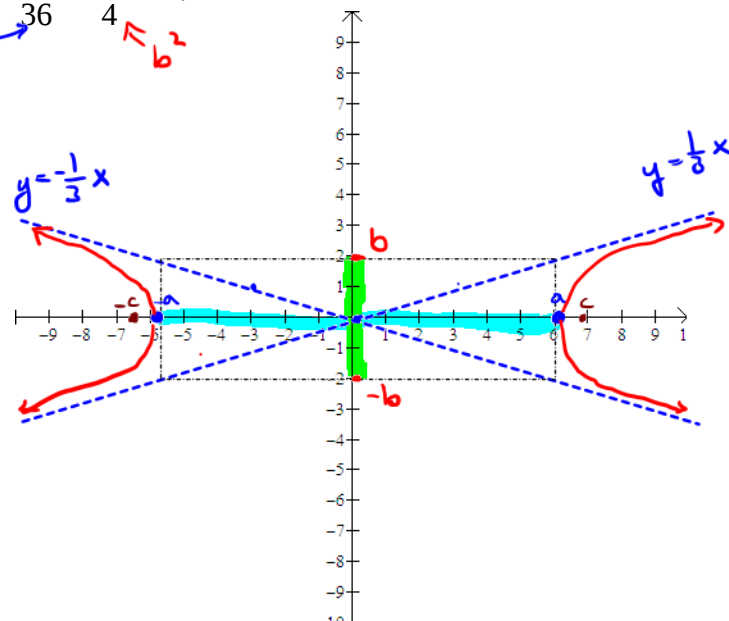
horizontal hyperbola with center (0,0)

Example 1: Find all relevant information and graph $\frac{x^2}{36} - \frac{y^2}{4} = 1$.

$$a^2 = 36 \Rightarrow a = 6$$

$$b^2 = 4 \Rightarrow b = 2$$

$$c^2 = a^2 + b^2 = 40 \Rightarrow c = \sqrt{40} = 2\sqrt{10}$$



Vertices: $(6, 0), (-6, 0)$

Foci: $(2\sqrt{10}, 0), (-2\sqrt{10}, 0)$

Eccentricity: $\frac{c}{a} = \frac{2\sqrt{10}}{6} =$

Transverse Axis: = The segment joining vertices

Length of transverse axis:

$$2 \cdot a = 2 \cdot 6 = 12$$

Conjugate axis: = The segment joining the "b" points.

Length of conjugate axis:

$$2 \cdot b = 2 \cdot 2 = 4$$

Slant Asymptotes:

$$y = \frac{b}{a}x = \frac{2}{6}x = \frac{1}{3}x$$

$$y = -\frac{b}{a}x = -\frac{1}{3}x$$

Example 2: Find all relevant information and graph

vertical hyperbola with center (0,0)

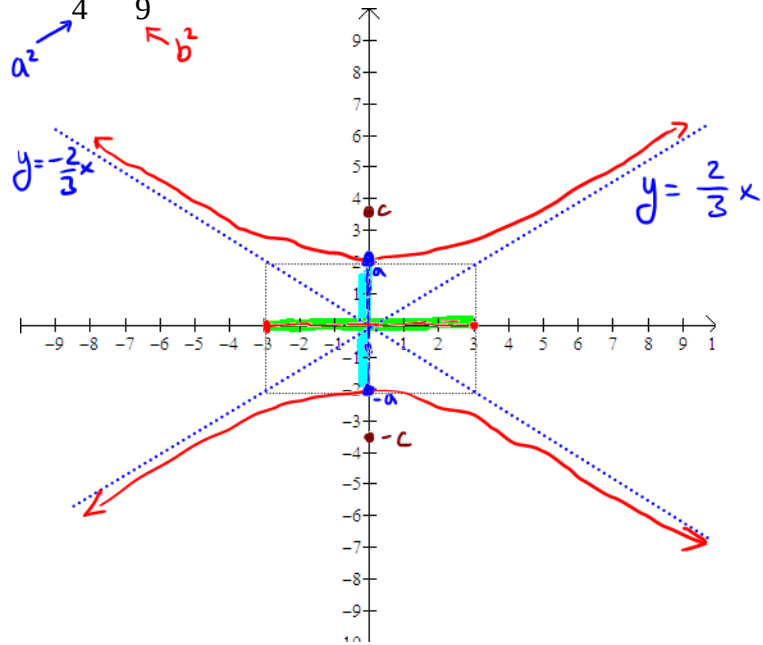
$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

$$a^2 = 4 \Rightarrow a = \pm 2 \text{ (vertices)}$$

$$b^2 = 9 \Rightarrow b = \pm 3$$

$$c^2 = a^2 + b^2 = 4 + 9 = 13$$

$$\Rightarrow c = \pm \sqrt{13}$$



Vertices: $(0, 2), (0, -2)$

Foci: $(0, \sqrt{13}), (0, -\sqrt{13})$

Eccentricity: $\frac{c}{a} = \frac{\sqrt{13}}{2}$

Transverse Axis: the segment joining vertices

Length of transverse axis: $2a = 2 \cdot 2 = 4$

Conjugate axis: the segment joining "b"

Length of conjugate axis: $2b = 2 \cdot 3 = 6$

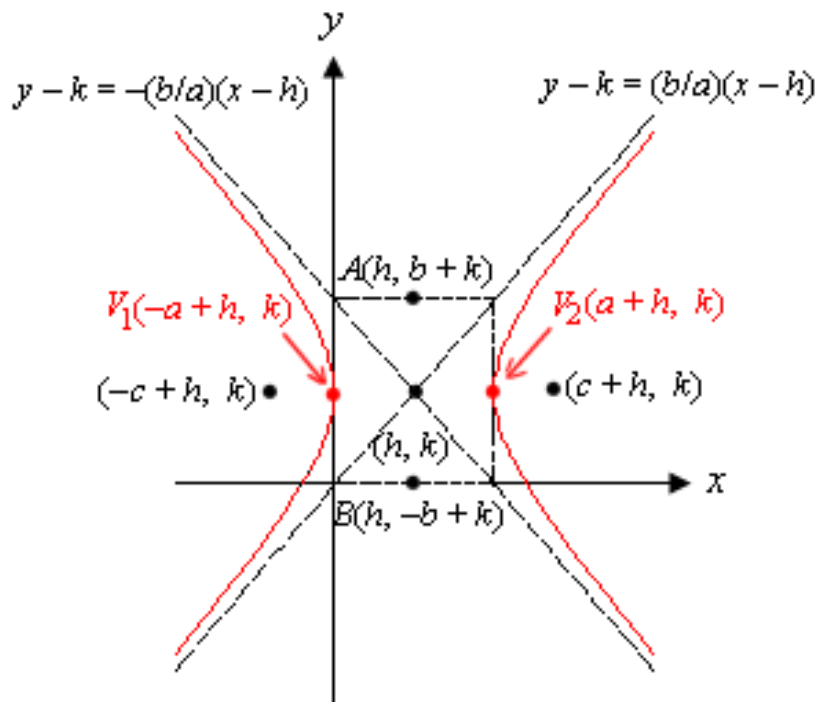
Slant Asymptotes: $y = \pm \frac{a}{b} x = \pm \frac{2}{3} x$

The equation of a hyperbola with center not at the origin: Center: (h, k)

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ or } \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

To graph a hyperbola with center not at the origin:

- Rearrange (complete the square if necessary) to look like $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$.
- Start at the center (h, k) and then graph it as before.
- To write down the equations of the asymptotes, start with the equations of the asymptotes for the similar hyperbola with center at the origin. Then replace x with x - h and replace y with y - k.



$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
shift h units horizontally
shift k units vertically
----->
 center (0,0)

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
 center (h, k)

- All the features shift accordingly.

The following list reflects the changes in translating the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to the

hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$:

Center: The point $(0, 0)$ changes to the point (h, k) .

Foci: The foci change from the points $(-c, 0)$ and $(c, 0)$ to the points $(-c+h, k)$ and $(c+h, k)$, where $c^2 = a^2 + b^2$.

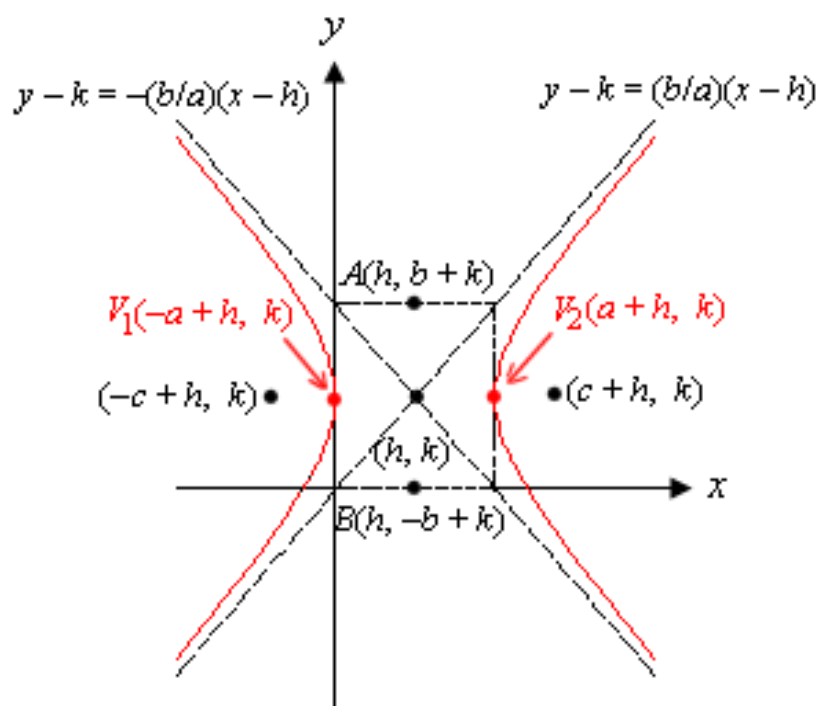
Vertices: The vertices change from the points $(-a, 0)$ and $(a, 0)$ to the points $(-a+h, k)$ and $(a+h, k)$.

Transverse Axis: $\overline{V_1V_2}$ Length of Transverse Axis: $2a$

Conjugate Axis: $\overline{AB}$ Length of Conjugate Axis: $2b$

Equations of the Asymptotes: The lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ change to the lines

$y - k = \frac{b}{a}(x - h)$ and $y - k = -\frac{b}{a}(x - h)$.



(look at the next page how to bring in standard form:

Example 3: Write the equation in standard form, find all relevant information and graph

$$9x^2 - 16y^2 - 18x + 96y = 279.$$

Standard form $\rightarrow \frac{(x-1)^2}{16} - \frac{(y-3)^2}{9} = 1$

$a^2 \rightarrow 16$ $b^2 \rightarrow 9$

Horizontal hyperbola with center (0,0) • Horizontal hyperbola shifted right, 3 up center (1,3) ✓

$a^2=16 \Rightarrow a=4$ • Vertices shifted
 $(-4,0), (4,0)$ $(-3,3), (5,3)$ ✓

$2a=2 \cdot 4=8$ • Transverse axis = $2a=8$

$c^2=a^2+b^2=25$ • Foci Coordinates shifted
 $c=5$
 $(-5,0), (5,0)$ $(-4,3), (6,3)$ ✓

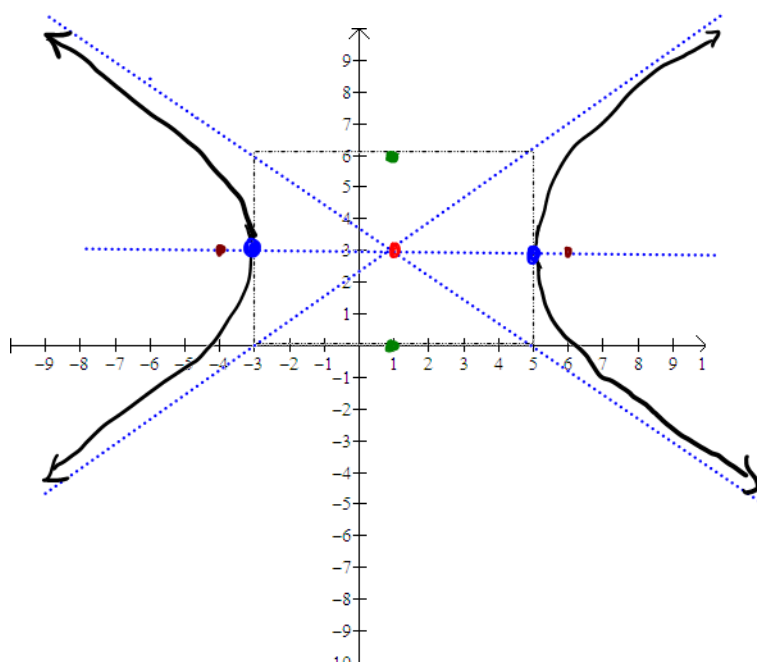
$b^2=9 \Rightarrow b=3$ • Conjugate axis = $2b=6$
 $2b=2 \cdot 3=6$
 $(0,-3), (0,3)$ shifted $(1,0), (1,6)$ ✓

$y = \frac{b}{a}x = \frac{3}{4}x$ • Slant Asymptotes shifted

$$y = -\frac{b}{a}x = -\frac{3}{4}x$$

$$y-3 = \frac{3}{4}(x-1), y-3 = -\frac{3}{4}(x-1)$$

$e = \frac{c}{a}$ • Eccentricity $c = \frac{c}{a} = \frac{5}{4} = 1.25$



$\Rightarrow$ Draw the rectangle.
 Then draw diagonals of rectangle and extend. Diagonals are the slant asymptotes

$$9x^2 - 16y^2 - 18x + 96y = 279$$

↳ Likely terms

$$(9x^2 - 18x) + (-16y^2 + 96y) = 279$$

↳ Factor coefficients and complete square

$$9(x^2 - 2x + 1) - 16(y^2 - 6y + 9) = 279 + 9 \cdot 1 - 16 \cdot 9$$

$(\frac{2}{2}=1)^2$
 $(\frac{6}{2}=3)^2$

↳ Rewrite

$$9(x-1)^2 - 16(y-3)^2 = 144$$

↳ Divide by 144 both sides

$$\frac{9(x-1)^2}{144} - \frac{16(y-3)^2}{144} = \frac{144}{144}$$

↳ Simplify

$$\boxed{\frac{(x-1)^2}{16} - \frac{(y-3)^2}{9} = 1}$$

Standard form.

ex. $\frac{(y-3)^2}{a^2} - \frac{(x+2)^2}{b^2} = 1$

$a^2 \rightarrow 4$ $b^2 \leftarrow 9$

$\hookrightarrow$ Vertical hyperbola

$\rightarrow y = \pm \frac{a}{b} x$

$\Rightarrow$ Find the slant asymptotes.

$$\begin{cases} a^2 = 4 \Rightarrow a = 2 \\ b^2 = 9 \Rightarrow b = 3 \end{cases}$$

$y = \pm \frac{2}{3} x$

$\xrightarrow{\hspace{1cm}}$
2 units left
3 units up

$y - 3 = \pm \frac{2}{3} (x + 2)$

Done

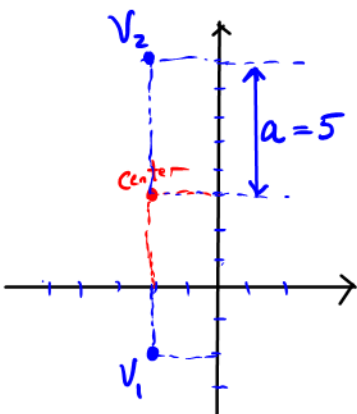
To be continued on Thursday, 09/24

- Keep in mind : Center, Vertices, Foci are on the same line, always.

Example 4: Write an equation of the hyperbola with center at $(-2, 3)$, one vertex is at $(-2, -2)$ and eccentricity is 2.

Center gives the shiftment of hyperbola

A quick sketch of problem:



- By picture, it's a vertical hyperbola centered @ $(-2, 3)$

- Vertices are symmetric wrt. center, hence

$$V_2(-2, 8), \text{ and } a=5. \Rightarrow a^2=25$$

- Need b , $e = \frac{c}{a} \Rightarrow c = e \cdot a = 2 \cdot 5 = 10$

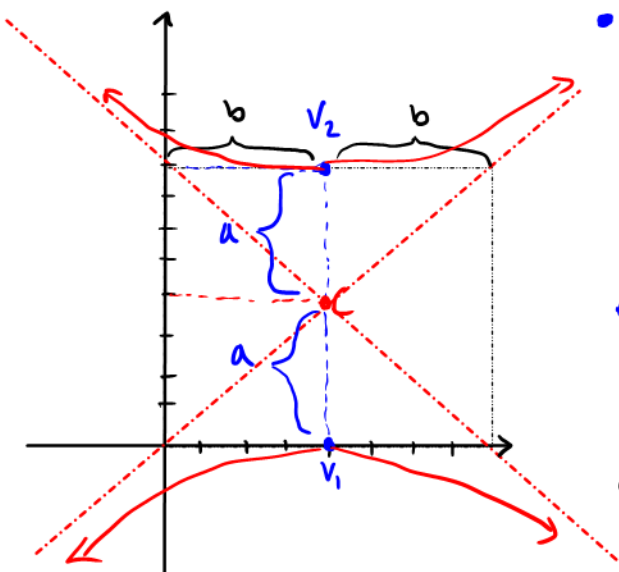
$$c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2 = 10^2 - 5^2 = 75$$

$$\Rightarrow \frac{(y-3)^2}{25} - \frac{(x+2)^2}{75} = 1$$

Exercise

Example 5: Write an equation of the hyperbola if the vertices are $(4, 0)$ and $(4, 8)$ and the asymptotes have slopes ± 1 .

$\underbrace{(4, 0)}_{V_1} \quad \underbrace{(4, 8)}_{V_2}$



- Vertices lined vertically $\Rightarrow$ Vertical hyperbola.

Center is the midpoint of transverse axis.

$$\Rightarrow C = (4, 4) \text{ shifts of hyperbola.}$$

- $a = \text{half of transverse length} = \frac{8}{2} = 4.$

$$a^2 = 16.$$

- Being vertical, slant asymptotes are $y = \pm \frac{a}{b}x$

$$\Rightarrow \frac{a}{b} = 1 \Rightarrow a = b \Rightarrow b^2 = 16$$

$$\Rightarrow \frac{(y-4)^2}{16} - \frac{(x-4)^2}{16} = 1$$