CHAPTER 1

Logic and Proofs

We recommend that you read the *Preface to the Student* before beginning this first chapter. Most of the terms and concepts in that *Preface* should be familiar to you, but it is well worth making sure you know the terminology and notations we will use throughout the book. It is especially important that you know precisely the definitions of such terms as: "divides," "prime," "rational," and "even" and "odd."

As described in the *Preface*, mathematics is concerned with the formation of a **theory** (collection of true statements) that describes patterns or relationships among quantities and structures. It is characterized by **deductive reasoning**, in which one uses logic to develop and extend a theory by drawing conclusions based on statements accepted as true. We give **proofs** to demonstrate that our conclusions are true. This chapter will provide a working knowledge of the basics of logic and how to construct a proof.

1.1 Propositions and Connectives

Our goal in this section is to understand truth values of propositions and how propositions can be combined using logical connectives.

Most sentences, such as "$\pi > 3$" and "Earth is the closest planet to the sun," have a truth value. That is, they are either true or false. We call these sentences propositions. Other sentences, such as "What time is it?" and "Look out!" are interrogatory or exclamatory; they express complete thoughts but have no truth value.

DEFINITION A **proposition** is a sentence that has exactly one truth value: true, which we denote by T, or false, which we denote by F.

Some propositions, such as "$7^2 = 60$," have easily determined truth values. It will take years to determine the truth value of the proposition "The North Pacific right whale will be an extinct species before the year 2525." Other statements, such as "Euclid was left-handed," are propositions whose truth values may never be known.

Sentences like "She lives in New York City" and "$x^2 = 36$" are not propositions because each could be true or false depending upon the person to whom "she" refers and what numerical value is assigned to x. We will deal with sentences like these in Section 1.3.

The statement "This sentence is false" is not a proposition because it is neither true nor false. It is an example of a **paradox**—a situation in which, from premises that look reasonable, one uses apparently acceptable reasoning to derive a conclusion that seems to be contradictory. If the statement "This sentence is false" is true, then by its meaning it must be false. On the other hand, if the given statement is false, then what it claims is false, so it must be true. The study of paradoxes such as this has played a key role in the development of modern mathematical logic. A famous example of a paradox formulated in 1901 by Bertand Russell* is discussed in Section 2.1.

By applying logical connectives to propositions, we can form new propositions.

DEFINITION The **negation** of a proposition P, denoted $\sim P$, is the proposition "not P." The proposition $\sim P$ is true exactly when P is false.

The truth value of the negation of a proposition is the opposite of the truth value of the proposition. For example, the negation of the false proposition "7 is divisible by 2" is the true statement "It is not the case that 7 is divisible by 2," or "7 is not divisible by 2."

DEFINITIONS Given propositions P and Q, the **conjunction** of P and Q, denoted $P \wedge Q$, is the proposition "P and Q." $P \wedge Q$ is true exactly when *both* P and Q are true.

The **disjunction** of P and Q, denoted $P \vee Q$, is the proposition "P or Q." $P \vee Q$ is true exactly when *at least one* of P or Q is true.

Examples. If C is the proposition "19 is composite" and M is "45 is a multiple of 3," we know C is false and M is true. Thus "19 is composite and 45 is a multiple of 3," written using logical connectives as $C \wedge M$, is a false proposition, while "19 is composite or 45 is a multiple of 3," which has form $C \vee M$, is true. The false proposition "Either 19 is composite or 45 is not a multiple of 3" has the form $C \vee \sim M$.

The English words *but*, *while*, and *although* are usually translated symbolically with the conjunction connective, because they have the same meaning as *and*. For

* Bertrand Russell (1872–1970) was a British philosopher, mathematician, and advocate for social reform. He was a strong voice for precision and clarity of arguments in mathematics and logic. He coauthored *Principia Mathematica* (1910–1913), a monumental effort to derive all of mathematics from a specific set of axioms and well-defined rules of inference.

example, we would write "19 is not composite, but 45 is a multiple of 3" in symbolic form as: $(\sim C) \wedge M$.

An important distinction must be made between a statement and the *form* of a statement. In the previous example "19 is composite and 45 is a multiple of 3" is a proposition with truth value F. We used the form $C \wedge M$ to represent this proposition, but *the form* $C \wedge M$ *itself has no truth value* unless C and M are assigned to be specific propositions. If we let C be "Copenhagen is the capital of Denmark" and M be "Madrid is the capital of Spain," then $C \wedge M$ would have the value T.

To repeat: a propositional form does not have *a* truth value. Instead, each form has a *list* of truth values that depend on the values assigned to its components. This list is displayed by presenting all possible combinations for the truth values of its components in a truth table. Since the connectives $\wedge$ and $\vee$ involve two components, their truth tables must list the four possible combinations of the truth values of those components:

P	Q	$P \wedge Q$
T	T	T
F	T	F
T	F	F
F	F	F

P	Q	$P \vee Q$
T	T	T
F	T	T
T	F	T
F	F	F

Since the value of $\sim P$ depends only on the two possible values for P, its truth table is

P	$\sim P$
T	F
F	T

Frequently you will encounter compound propositions formed from more than two propositional variables. The propositional form $(P \wedge Q) \vee \sim R$ has three variables P, Q, and R; it follows that there are $2^3 = 8$ possible combinations of truth values. The two main components are $P \wedge Q$ and $\sim R$. We make truth tables for these and combine them by using the truth table for $\vee$.

P	Q	R	$P \wedge Q$	$\sim R$	$(P \wedge Q) \vee \sim R$
T	T	T	T	F	T
F	T	T	F	F	F
T	F	T	F	F	F
F	F	T	F	F	F
T	T	F	T	T	T
F	T	F	F	T	T
T	F	F	F	T	T
F	F	F	F	T	T

The statement "Either 7 is prime and 9 is even or else 11 is not less than 3" may be symbolized by $(P \wedge Q) \vee \sim R$, where P is "7 is prime," Q is "9 is even," and R

is "11 is less than 3." We know P is true, Q is false and R is false. Therefore, $(P \wedge Q)$ is false and $\sim R$ is true. Thus $(P \wedge Q) \vee \sim R$ is true, in agreement with line 7 of the table. Thus the proposition "Either 7 is prime and 9 is even or else 11 is not less than 3" is a true statement.

Some compound forms always yield the value true just because of the way they are formed; others are always false.

DEFINITIONS A **tautology** is a propositional form that is true for every assignment of truth values to its components.

A **contradiction** is a propositional form that is false for every assignment of truth values to its components.

For example, the *Law of Excluded Middle*, $P \vee \sim P$, is a tautology because $P \vee \sim P$ is true when P is true and true when P is false. We know that a statement like "The absolute value function is continuous or it is not continuous" must be true because it has the form of the Law of Excluded Middle.

Example. Show that $(P \vee Q) \vee (\sim P \wedge \sim Q)$ is a tautology.

The truth table for this propositional form is

P	Q	$P \vee Q$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$(P \vee Q) \vee (\sim P \wedge \sim Q)$
T	T	T	F	F	F	T
F	T	T	T	F	F	T
T	F	T	F	T	F	T
F	F	F	T	T	T	T

Since the last column is all true, $(P \vee Q) \vee (\sim P \wedge \sim Q)$ is a tautology.

Both $\sim(P \vee \sim P)$ and $Q \wedge \sim Q$ are examples of contradictions. The negation of a contradiction is, of course, a tautology.

Writing a proof requires the ability to connect statements so that the truth of any given statement in the proof follows logically from previous statements in the proof, from known results, or from basic assumptions. Particularly important is the ability to recognize or write a statement equivalent to another. Sometimes, it is the *form* of a compound statement that may be used to find a useful equivalent.

DEFINITION Two propositional forms are **equivalent** if and only if they have the same truth tables.

Example. The propositional forms P and $\sim(\sim P)$ are equivalent. The truth tables for these forms may be combined in one table to show that they are the same:

P	$\sim P$	$\sim(\sim P)$
T	F	T
F	T	F

The fact that P and $\sim(\sim P)$ have the same truth value for each line of the truth table means that whatever proposition we choose for P, the truth value of P and $\sim(\sim P)$ are identical.

Some of the most commonly used equivalent forms are presented in the following theorem.

Theorem 1.1.1 For propositions P, Q, and R, the following are equivalent:

(a)	P	and	$\sim(\sim P)$	Double Negation Law
(b)	$P \vee Q$	and	$Q \vee P$	Commutative Laws
(c)	$P \wedge Q$	and	$Q \wedge P$	
(d)	$P \vee (Q \vee R)$	and	$(P \vee Q) \vee R$	Associative Laws
(e)	$P \wedge (Q \wedge R)$	and	$(P \wedge Q) \wedge R$	
(f)	$P \wedge (Q \vee R)$	and	$(P \wedge Q) \vee (P \wedge R)$	Distributive Laws
(g)	$P \vee (Q \wedge R)$	and	$(P \vee Q) \wedge (P \vee R)$	
(h)	$\sim(P \wedge Q)$	and	$\sim P \vee \sim Q$	DeMorgan's* Laws
(i)	$\sim(P \vee Q)$	and	$\sim P \wedge \sim Q$	

Proof.

(a) See the discussion above.

(h) By examining the fourth and seventh columns of their combined truth tables as shown here,

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$
T	T	T	F	F	F	F
F	T	F	T	T	F	T
T	F	F	T	F	T	T
F	F	F	T	T	T	T

we see that the truth tables for $\sim(P \wedge Q)$ and $\sim P \vee \sim Q$ are identical. Thus $\sim(P \wedge Q)$ and $\sim P \vee \sim Q$ are equivalent propositional forms.

Proofs of the remaining parts are left as exercises. ■

In addition to making tables to verify the remaining parts of Theorem 1.1.1, you should also think about why two propositional forms are equivalent by looking

* Augustus DeMorgan (1806–1871) was an English logician and mathematician whose contributions include his notational system for symbolic logic. He also introduced the term "mathematical induction" (see Section 2.4) and developed a rigorous foundation for that proof technique.

at their meanings. For part (h), negation is applied to a conjunction. The form $\sim(P \wedge Q)$ is true precisely when $P \wedge Q$ is false. This happens when one of P or Q is false, or in other words, when one of $\sim P$ or $\sim Q$ is true. Thus, $\sim(P \wedge Q)$ is equivalent to $\sim P \vee \sim Q$. That is, to say "You don't have both P and Q" is the same as saying "You don't have P or you don't have Q."

As an example of how this theorem might be useful in dealing with statements, suppose we are told that the statement "The function f is increasing and concave upward" is false. The statement has the form $P \wedge Q$, where P is the statement "f is increasing" and Q is the statement "f is concave upward." The negation $\sim(P \wedge Q)$ is "It is not the case that f is increasing and f is concave upward." By part (h) above, this is equivalent to $\sim P \vee \sim Q$, which is

> "It is not the case that f is increasing or it is not the case that f is concave upward."

An easier way to say this is

"f is not increasing or f is not concave upward."

A **denial** of a proposition P is any proposition equivalent to $\sim P$. A proposition has only one negation, $\sim P$, but always has many denials, including $\sim P$, $\sim\sim\sim P$, $\sim\sim\sim\sim\sim P$, etc. DeMorgan's Laws provide others ways to construct useful denials.

Example. A denial of "Either Miss Scarlet is not guilty or the crime did not take place in the ballroom" is "The crime took place in the ballroom and Miss Scarlet is guilty." This can be verified by writing the two propositions symbolically as $(\sim S) \vee (\sim B)$ and $B \wedge S$, respectively, and checking that their truth tables have exactly opposite values. We could also observe that $B \wedge S$ is equivalent to $S \wedge B$ so a denial of $B \wedge S$ is equivalent to $\sim(S \wedge B)$, which we know by DeMorgan's Laws is equivalent to $(\sim S) \vee (\sim B)$.

Example. The statement "Line L_1 has slope 3/5 or line L_2 does not have slope -4" may be symbolized using the form $P \vee \sim Q$, so its negation is $\sim(P \vee \sim Q)$. We can write a simpler denial $(\sim P) \wedge Q$ by applying DeMorgan's Laws and the Double Negation Law. The simplified denial says "Line L_1 does not have slope $3/5$ and line L_2 has slope -4."

Notice that someone might read the negation $\sim(P \vee \sim Q)$ as "It is not the case that L_1 has slope $3/5$ or line L_2 does not have slope -4." This sentence is ambiguous because without some further explanation, it is not clear if the phrase "It is not the case" refers to the entire remainder of the sentence or to just "L_1 has slope $3/5$."

Ambiguities like the one above are sometimes allowable in English but can cause trouble in mathematics. To avoid ambiguities, you should use delimiters, such as parentheses ( ), square brackets [ ], and braces { }.

To avoid writing large numbers of delimiters, we use the following rules, which we refer to as the *hierarchy of connectives*.

First, $\sim$ always is applied to the smallest proposition following it.
Then, $\wedge$ always connects the smallest propositions surrounding it.
Finally, $\vee$ connects the smallest propositions surrounding it.

Thus, $\sim P \vee Q$ is an abbreviation for $(\sim P) \vee Q$, but $\sim(P \vee Q)$ is the only way to write the negation of $P \vee Q$. Here are some other examples:

$P \vee Q \wedge R$ abbreviates $P \vee (Q \wedge R)$.
$P \wedge \sim Q \vee \sim R$ abbreviates $[P \wedge (\sim Q)] \vee (\sim R)$.
$\sim P \vee \sim Q$ abbreviates $(\sim P) \vee (\sim Q)$.
$\sim P \wedge \sim R \vee \sim P \wedge R$ abbreviates $[(\sim P) \wedge (\sim R)] \vee [(\sim P) \wedge R]$.

When the same connective is used several times in succession, parentheses may be omitted. We reinsert parentheses from the left, so that $P \vee Q \vee R$ is really $(P \vee Q) \vee R$. For example, $R \wedge P \wedge \sim P \wedge Q$ abbreviates $[(R \wedge P) \wedge (\sim P)] \wedge Q$, whereas $R \vee P \wedge \sim P \vee Q$, which does not use the same connective consecutively, abbreviates $(R \vee [P \wedge (\sim P)]) \vee Q$. Leaving out parentheses is not required; some propositional forms are much easier to read with a few well-chosen "unnecessary" parentheses.

Exercises 1.1

1. Use your knowledge of number systems to determine whether each is true or false.

- **(a)** 11 is a rational number.
- ★ **(b)** 5π is a rational number.
- **(c)** There are exactly 3 prime numbers between 40 and 50.
- **(d)** There are exactly 5 prime numbers less than 10.
- **(e)** 29 is a composite number.
- **(f)** 0 is a natural number.
- ★ **(g)** $(5 + 2i)(5 - 2i)$ is a real number.
- **(h)** 18 is a multiple of 12.

2. Which of the following are propositions? Give the truth value of each proposition.

- **(a)** What time is dinner?
- **(b)** It is not the case that $5 + \pi$ is not a rational number.
- ★ **(c)** $x/2$ is a rational number.
- **(d)** $2x + 3y$ is a real number.
- **(e)** Either $3 + \pi$ is rational or $3 - \pi$ is rational.
- ★ **(f)** Either 2 is rational and π is irrational, or 2π is rational.
- **(g)** Either 5π is rational and 4.9 is rational, or 3π is rational.
- **(h)** $-\frac{1}{2}$ is rational, and either $3\pi < 10$ or $3\pi > 15$.
- **(i)** It is not the case that 39 is prime, or that 64 is a power of 2.
- **(j)** There are more than three false statements in this book and this statement is one of them.

3. Make truth tables for each of the following propositional forms.

- ★ **(a)** $P \wedge \sim P$.
- **(b)** $P \vee \sim P$.
- ★ **(c)** $P \wedge \sim Q$.
- **(d)** $P \wedge (Q \vee \sim Q)$.
- ★ **(e)** $(P \wedge Q) \vee \sim Q$.
- **(f)** $\sim(P \wedge Q)$.
- **(g)** $(P \vee \sim Q) \wedge R$.
- **(h)** $\sim P \wedge \sim Q$.

★ **(i)** $P \wedge (Q \vee R)$. **(j)** $(P \wedge Q) \vee (P \wedge R)$.
(k) $P \wedge P$. **(l)** $(P \wedge Q) \vee (R \wedge \sim S)$.

4. If P, Q, and R are true while S and T are false, which of the following are true?
★ **(a)** $Q \wedge (R \wedge S)$. **(b)** $Q \vee (R \wedge S)$.
★ **(c)** $(P \vee Q) \wedge (R \vee S)$. **(d)** $(\sim P \vee \sim Q) \vee (\sim R \vee \sim S)$.
(e) $\sim P \vee \sim Q$. ★ **(f)** $(\sim Q \vee S) \wedge (Q \vee S)$.
★ **(g)** $(P \vee S) \wedge (P \vee T)$.

5. Use truth tables to prove the remaining parts of Theorem 1.1.1.

6. Which of the following pairs of propositional forms are equivalent?
★ **(a)** $P \wedge P, P$. **(b)** $P \vee P, P$.
★ **(c)** $P \wedge Q, Q \wedge P$. **(d)** $(\sim P) \vee (\sim Q), \sim (P \vee \sim Q)$.
★ **(e)** $\sim P \wedge \sim Q, \sim (P \wedge \sim Q)$. **(f)** $\sim (P \wedge Q), \sim P \wedge \sim Q$.
★ **(g)** $(P \wedge Q) \vee R, P \wedge (Q \vee R)$. **(h)** $(P \wedge Q) \vee R, P \vee (Q \wedge R)$.

7. Determine the propositional form and truth value for each of the following:
(a) It is not the case that 2 is odd.
(b) $f(x) = e^x$ is increasing and concave up.
(c) Both 7 and 5 are factors of 70.
(d) Perth or Panama City or Pisa is located in Europe.

8. P, Q, and R are propositional forms, and P is equivalent to Q, and Q is equivalent to R. Prove that
★ **(a)** Q is equivalent to P.
(b) P is equivalent to R.
(c) $\sim Q$ is equivalent to $\sim P$.

9. Use a truth table to determine whether each of the following is a tautology, a contradiction, or neither.
(a) $(P \wedge Q) \vee (\sim P \wedge \sim Q)$.
(b) $\sim (P \wedge \sim P)$.
★ **(c)** $(P \wedge Q) \vee (\sim P \vee \sim Q)$.
(d) $(A \wedge B) \vee (A \wedge \sim B) \vee (\sim A \wedge B) \vee (\sim A \wedge \sim B)$.
(e) $(Q \wedge \sim P) \wedge \sim (P \wedge R)$.
(f) $P \vee [(\sim Q \wedge P) \wedge (R \vee Q)]$.

10. Suppose A is a tautology and B is a contradiction. Are the following tautologies, contradictions, or neither?
★ **(a)** $A \wedge B$. **(b)** $A \wedge \sim B$.
★ **(c)** $A \vee B$. **(d)** $\sim (\sim A \wedge B)$.

11. Give a useful denial of each statement.
★ **(a)** x is a positive integer. (Assume that x is some fixed integer.)
(b) Cleveland will win the first game or the second game.
★ **(c)** $5 \geq 3$.
(d) 641,371 is a composite integer.
★ **(e)** Roses are red and violets are blue.
(f) T is not bounded or T is compact. (Assume that T is a fixed object.)
(g) M is odd and one-to-one. (Assume that M is some fixed function.)

(h) The function f has positive first and second derivatives at x_0. (Assume that f is a fixed function and x_0 is a fixed real number.)

(i) The function g has a relative maximum at $x = 2$ or $x = 4$ and a relative minimum at $x = 3$. (Assume that g is a fixed function.)

(j) Neither $z < s$ nor $z \leq t$ is true. (Assume that z, s, and t are fixed real numbers.)

(k) R is transitive but not reflexive. (Assume that R is a fixed object.)

12. Restore parentheses to these abbreviated propositional forms.

(a) $\sim\sim P \vee \sim Q \wedge \sim S$.

(b) $Q \wedge \sim S \vee \sim(\sim P \wedge Q)$.

(c) $P \wedge \sim Q \;\vee \sim P \wedge \sim R \vee \sim P \wedge S$.

(d) $\sim P \vee Q \wedge \sim\sim P \wedge Q \vee R$.

13. Other logical connectives between two propositions P and Q are possible.

(a) The word *or* is used in two different ways in English. We have presented the truth table for $\vee$, the **inclusive or,** whose meaning is "one or the other or both." The **exclusive or,** meaning "one or the other but not both" and denoted $\circledvee$, has its uses in English, as in "She will marry Heckle or she will marry Jeckle." The "inclusive or" is much more useful in mathematics and is the accepted meaning unless there is a statement to the contrary.

★ **(i)** Make a truth table for the "exclusive or" connective $\circledvee$.

(ii) Show that $A \circledvee B$ is equivalent to $(A \vee B) \wedge \sim(A \wedge B)$.

(b) "NAND" and "NOR" circuits are commonly used as a basis for flash memory chips. A NAND B is defined to be the negation of "A and B." A NOR B is defined to be the negation of "A or B."

(i) Write truth tables for NAND and NOR connectives.

(ii) Show that (A NAND B) $\vee$ (A NOR B) is equivalent to (A NAND B).

(iii) Show that (A NAND B) $\wedge$ (A NOR B) is equivalent to (A NOR B).

1.2 Conditionals and Biconditionals

Sentences of the form "If P, then Q" are the most important kind of propositions in mathematics. You have seen many examples of such statements in mathematics courses: from precalculus, "If two lines in a plane have the same slope, then the lines are parallel"; from trigonometry, "If $\sec\theta = \frac{5}{3}$, then $\sin\theta = \frac{4}{5}$."; from calculus, "If f is differentiable at x_0 and $f(x_0)$ is a relative minimum for f, then $f'(x_0) = 0$."

DEFINITIONS For propositions P and Q, the **conditional sentence** $P \Rightarrow Q$ is the proposition "If P, then Q." Proposition P is called the **antecedent** and Q is the **consequent.** The conditional sentence $P \Rightarrow Q$ is true if and only if P is false or Q is true.

The truth table for $P \Rightarrow Q$ is

P	Q	$P \Rightarrow Q$
T	T	T
F	T	T
T	F	F
F	F	T

According to this table, there is only one way that $P \Rightarrow Q$ can be false: when P is true and Q is false. Thus, this truth table agrees with the way we understand promises: the only situation where a promise is broken is when the antecedent is true but the person making the promise fails to make the consequent true.

Example. Suppose someone says to a friend "If the concert is sold out, I'll take you sailing." This promise is broken (the conditional sentence is false) only when the concert was sold out (the antecedent is true) and the person who made the promise did not take the other person sailing (the consequent is false). This is line 3 of the truth table. In all other situations, the promise is true. If there were tickets left (lines 2 and 4 of the table), we don't say the promise was broken, regardless of whether the friends decided to go sailing. The promise is also kept in the situation where the concert is sold out and the friends went sailing, which is line 1 of the table.

One curious consequence of the truth table for $P \Rightarrow Q$ is that a conditional sentence may be true even when there is no connection between the antecedent and the consequent. The reason for this is that the truth value of $P \Rightarrow Q$ depends *only* on the truth value of components P and Q, not on their interpretation. For this reason all of the following are true:

"If $\sin \pi = 1$, then 6 is prime." (line 4 of the truth table)
"$13 > 7 \Rightarrow 2 + 3 = 5$." (line 1 of the truth table)
"$\pi = 3 \Rightarrow$ Paris is the capital of France." (line 2 of the truth table)

and both of these are false by line 3 of the truth table:

"If Saturn has rings, then $(2 + 3)^2 = 2^2 + 3^2$."
"If $4\pi > 10$, then 1 is a prime number."

Other consequences of the truth table for $P \Rightarrow Q$ are worth noting. When P is false, it doesn't matter what truth value Q has: $P \Rightarrow Q$ will be true by lines 2 and 4. When Q is true, it doesn't matter what truth value P has: $P \Rightarrow Q$ will be true by lines 1 and 2. Finally, when P and $P \Rightarrow Q$ are both true (on line 1), Q must also be true.

Example. Both propositions

"If Isaac Newton was born in 1642, then $3 \cdot 5 = 15$"
"If Isaac Newton was born in 1643, then $3 \cdot 5 = 15$"

are true because the consequent "$3 \cdot 5 = 15$" is true.

Our truth table definition for $P \Rightarrow Q$ captures the same meaning for "If $\ldots$, then $\ldots$" that you have always used in mathematics. For example, if we think of x as some fixed real number, we all know that

$$\text{“If } x > 8, \text{ then } x > 5\text{”}$$

is a true statement, no matter what number x we have in mind. Let's examine why we say this sentence is true for some specific values of x, where the antecedent P is "$x > 8$" and the consequent Q is "$x > 5$."

In the case $x = 11$, both P and Q are true, as in line 1 of the truth table. The case $x = 7$ corresponds to the second line of the table, and for $x = 3$ we have the situation in line 4. There is no case corresponding to line 3 because $P \Rightarrow Q$ is true. Note that when we say "If P, then Q" is true, we don't claim that either P or Q is true. What we do say is that no matter what number we think of, *if* it's larger than 8, it's also larger than 5.

Two propositions closely related to $P \Rightarrow Q$ are its converse and contrapositive.

DEFINITION Let P and Q be propositions.

The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
The **contrapositive** of $P \Rightarrow Q$ is $(\sim Q) \Rightarrow (\sim P)$.

For the conditional sentence "If π is an integer, then 14 is even," the converse of the sentence is "If 14 is even, then π is an integer" and the contrapositive is "If 14 is not even, then π is not an integer." The converse is false, but the sentence and its contrapositive are true.

For the sentence "If $1 + 1 = 2$, then $\sqrt{10} > 3$," the converse and contrapositive are, respectively, "If $\sqrt{10} > 3$, then $1 + 1 = 2$" and "If $\sqrt{10}$ is not greater than 3, then $1 + 1$ is not equal to 2." In this example, all three sentences are true.

The previous two examples suggest that the truth values of a conditional sentence and its contrapositive are related, but there seems to be little connection between the truth values of $P \Rightarrow Q$ and its converse. We describe the relationships in the following theorem.

Theorem 1.2.1 For propositions P and Q,

(a) $P \Rightarrow Q$ is equivalent to its contrapositive $(\sim Q) \Rightarrow (\sim P)$.
(b) $P \Rightarrow Q$ is *not* equivalent to its converse $Q \Rightarrow P$.

Proof. The proofs are carried out by examination of the truth tables.

P	Q	$P \Rightarrow Q$	$\sim P$	$\sim Q$	$(\sim Q) \Rightarrow (\sim P)$	$Q \Rightarrow P$
T	T	T	F	F	T	T
F	T	T	T	F	T	F
T	F	F	F	T	F	T
F	F	T	T	T	T	T

(a) $P \Rightarrow Q$ is equivalent to $(\sim Q) \Rightarrow (\sim P)$ because the third column in the truth table is identical to the sixth column in the table.

(b) $P \Rightarrow Q$ is not equivalent to $Q \Rightarrow P$ because column 3 in the truth table differs from column 7 in rows 2 and 3. ■

We have seen cases where a conditional sentence and its converse have the same truth value. Theorem 1.2.1(b) simply says that this need not always be the case—the truth values of $P \Rightarrow Q$ cannot be inferred from its converse $Q \Rightarrow P$.

The next connective we need is the biconditional connective $\Leftrightarrow$. The double arrow $\Leftrightarrow$ reminds one of both $\Leftarrow$ and $\Rightarrow$, and this is no accident, because $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

DEFINITION For propositions P and Q, the **biconditional sentence** $P \Leftrightarrow Q$ is the proposition "P if and only if Q." $P \Leftrightarrow Q$ is true exactly when P and Q have the same truth values. We also write P iff Q to abbreviate P if and only if Q.

The truth table for $P \Leftrightarrow Q$ is

P	Q	$P \Leftrightarrow Q$
T	T	T
F	T	F
T	F	F
F	F	T

Examples. The proposition "$2^3 = 8$ iff 49 is a perfect square" is true because both components are true. The proposition "$\pi = 22/7$ iff $\sqrt{2}$ is a rational number" is true because both components are false. The proposition "$6 + 1 = 7$ iff Lake Michigan is in Kansas" is false because the truth values of the components differ.

Definitions, fully stated with the "if and only if" connective, are important examples of biconditional sentences because they describe exactly the condition(s) to meet the definition. Although sometimes a definition does not explicitly use the iff wording, biconditionality does provide a good test of whether a statement could serve as a definition or just a description.

Example. The statement "Vertical lines have undefined slope" could be used as a definition because a line is vertical iff its slope is undefined. However, "A zebra is a striped animal" is not a definition, because the sentence "An animal is a zebra iff the animal is striped" is false.

Because the biconditional sentence $P \Leftrightarrow Q$ is true exactly when the truth values of P and Q agree, we can use the biconditional connective to restate the meaning of equivalent propositional forms:

The propositional forms P and Q are equivalent precisely when $P \Leftrightarrow Q$ is a tautology.

Thus each statement in Theorem 1.1.1 may be restated using the $\Leftrightarrow$ connective. For example, DeMorgan's Laws are:

$$\sim(P \wedge Q) \Leftrightarrow (\sim P \vee \sim Q) \text{ and}$$
$$\sim(P \vee Q) \Leftrightarrow (\sim P \wedge \sim Q).$$

All of the statements in Theorem 1.1.1 are used regularly in proofs. The next theorem contains several additional important pairs of equivalent propositional forms that involve implication. They, too, will be used often.

Theorem 1.2.2 For propositions P, Q, and R,

(a) $P \Rightarrow Q$ is equivalent to $\sim P \vee Q$.
(b) $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.
(c) $\sim(P \Rightarrow Q)$ is equivalent to $P \wedge \sim Q$.
(d) $\sim(P \wedge Q)$ is equivalent to $P \Rightarrow \sim Q$ and to $Q \Rightarrow \sim P$.
(e) $P \Rightarrow (Q \Rightarrow R)$ is equivalent to $(P \wedge Q) \Rightarrow R$.
(f) $P \Rightarrow (Q \wedge R)$ is equivalent to $(P \Rightarrow Q) \wedge (P \Rightarrow R)$.
(g) $(P \vee Q) \Rightarrow R$ is equivalent to $(P \Rightarrow R) \wedge (Q \Rightarrow R)$.

Exercise 8 asks you to prove each part of Theorem 1.2.2. The natural way to proceed is by constructing and then comparing truth tables, but you should also think about the meaning of both sides of each statement of equivalence. With part (a), for example, we reason as follows: $P \Rightarrow Q$ is false exactly when P is true and Q is false, which happens exactly when both $\sim P$ and Q are false. Since this happens exactly when $\sim P \vee Q$ is false, the truth tables for $P \Rightarrow Q$ and $\sim P \vee Q$ are identical.

Note that many of the statements in Theorems 1.1.1 and 1.2.2 are related. For example, once we have established Theorem 1.1.1 and 1.2.2(a), we reason that part (c) is correct as follows:

$\sim(P \Rightarrow Q)$ is equivalent, by part (a), to
$\sim(\sim P \vee Q)$, which is equivalent, by Theorem 1.1.1(i), to
$\sim(\sim P) \wedge \sim Q$, which is equivalent, by Theorem 1.1.1(a), to
$P \wedge \sim Q$.

Recognizing the structure of a sentence and translating the sentence into symbolic form using logical connectives are aids in determining its truth or falsity. The translation of sentences into propositional symbols is sometimes very complicated because some natural languages such as English are rich and powerful with many nuances. The ambiguities that we tolerate in English would destroy structure and usefulness if we allowed them in mathematics.

Even the translations of simple sentences can present special problems. Suppose a teacher says to a student

"If you score 74% or higher on the next test, you will pass this course."

This sentence clearly has the form of a conditional sentence, although almost everyone will interpret the meaning as a biconditional.

Contrast this with the situation in mathematics where "If $x = 2$, then x is a solution to $x^2 = 2x$" must have only the meaning of the connective $\Rightarrow$, because $x^2 = 2x$ does not imply $x = 2$.

Shown below are some phrases in English that are ordinarily translated by using the connectives $\Rightarrow$ or $\Leftrightarrow$. In the accompanying examples, think of a and t as fixed real numbers.

Use $P \Rightarrow Q$ to translate:	Examples:
If P, then Q.	If $a > 5$, then $a > 3$.
P implies Q.	$a > 5$ implies $a > 3$.
P is sufficient for Q.	$a > 5$ is sufficient for $a > 3$.
P only if Q.	$a > 5$ only if $a > 3$.
Q, if P.	$a > 3$, if $a > 5$.
Q whenever P.	$a > 3$ whenever $a > 5$.
Q is necessary for P.	$a > 3$ is necessary for $a > 5$.
Q, when P.	$a > 3$, when $a > 5$.

Use $P \Leftrightarrow Q$ to translate:	Examples:
P if and only if Q.	$\|t\| = 2$ if and only if $t^2 = 4$.
P if, but only if, Q.	$\|t\| = 2$ if, but only if, $t^2 = 4$.
P is equivalent to Q.	$\|t\| = 2$ is equivalent to $t^2 = 4$.
P is necessary and sufficient for Q.	$\|t\| = 2$ is necessary and sufficient for $t^2 = 4$.

The word *unless* is one of those connective words in English that poses special problems because it has so many different interpretations. See Exercise 11.

Examples. In these sentence translations, we assume that S, G, and e have been specified. It is not necessary to know the meanings of all the words because the form of the sentence is sufficient to determine the correct translation.

"S is compact is sufficient for S to be bounded" is translated

$$S \text{ is compact} \Rightarrow S \text{ is bounded.}$$

"A necessary condition for a group G to be cyclic is that G is abelian" is translated

$$G \text{ is cyclic} \Rightarrow G \text{ is abelian.}$$

"A set S is infinite if S has an uncountable subset" is translated

$$S \text{ has an uncountable subset} \Rightarrow S \text{ is infinite.}$$

"A necessary and sufficient condition for the graph G to be a tree is that G is connected and every edge of G is a bridge" is translated

$$G \text{ is a tree} \Leftrightarrow (G \text{ is connected} \wedge \text{every edge of } G \text{ is a bridge}).$$

Example. If we let P denote the proposition "Roses are red" and Q denote the proposition "Violets are blue," we can translate the sentence "It is not the case that roses are red, nor that violets are blue" in at least two ways: $\sim(P \vee Q)$ or $(\sim P) \wedge (\sim Q)$. Fortunately, these are equivalent by Theorem 1.1.1(h). Note that the proposition "Violets are purple" requires a new symbol, say R, since it expresses a new idea that cannot be formed from the components P and Q.

The sentence "17 and 35 have no common divisors" shows that the meaning, and not just the form of the sentence, must be considered in translating; it cannot be broken up into the two propositions: "17 has no common divisors" and "35 has no common divisors." Compare this with the proposition "17 and 35 have digits totaling 8," which can be written as a conjunction.

Example. Suppose b is a fixed real number. The form of the sentence "If b is an integer, then b is either even or odd" is $P \Rightarrow (Q \vee R)$, where P is "b is an integer," Q is "b is even," and R is "b is odd."

Example. Suppose a, b, and p are fixed integers. "If p is a prime number that divides ab, then p divides a or b" has the form $(P \wedge Q) \Rightarrow (R \vee S)$, where P is "p is a prime," Q is "p divides ab," R is "p divides a," and S is "p divides b."

The hierarchy of connectives in Section 1.1 that governs the use of parentheses for propositional forms can be extended to the connectives $\Rightarrow$ and $\Leftrightarrow$:

The connectives $\sim$, $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$ are always applied in the order listed.

Thus, $\sim$ applies to the smallest possible proposition, then $\wedge$ is applied with the next smallest scope, and so forth. For example,

$$P \Rightarrow \sim Q \vee R \Leftrightarrow S \text{ is an abbreviation for } (P \Rightarrow [(\sim Q) \vee R]) \Leftrightarrow S,$$
$$P \vee \sim Q \Leftrightarrow R \Rightarrow S \text{ is an abbreviation for } [P \vee (\sim Q)] \Leftrightarrow (R \Rightarrow S),$$

and

$$P \Rightarrow Q \Rightarrow R \text{ is an abbreviation for } (P \Rightarrow Q) \Rightarrow R.$$

Exercises 1.2

1. Identify the antecedent and the consequent for each of the following conditional sentences. Assume that a, b, and f represent some fixed sequence, integer, or function, respectively.

★ **(a)** If squares have three sides, then triangles have four sides.
(b) If the moon is made of cheese, then 8 is an irrational number.
(c) b divides 3 only if b divides 9.
★ **(d)** The differentiability of f is sufficient for f to be continuous.
(e) A sequence a is bounded whenever a is convergent.
★ **(f)** A function f is bounded if f is integrable.
(g) $1 + 2 = 3$ is necessary for $1 + 1 = 2$.

(h) The fish bite only when the moon is full.

★ (i) A time of 3 minutes, 48 seconds or less is necessary to qualify for the Olympic team.

☆ 2. Write the converse and contrapositive of each conditional sentence in Exercise 1.

3. What can be said about the truth value of Q when
(a) P is false and $P \Rightarrow Q$ is true?
(b) P is true and $P \Rightarrow Q$ is true?
(c) P is true and $P \Rightarrow Q$ is false?
(d) P is false and $P \Leftrightarrow Q$ is true?
(e) P is true and $P \Leftrightarrow Q$ is false?

4. Identify the antecedent and consequent for each conditional sentence in the following statements from this book.
(a) Theorem 1.3.1(a)
(b) Exercise 3 of Section 1.6
(c) Theorem 2.1.4
(d) The PMI, Section 2.4
(e) Theorem 2.6.4
(f) Theorem 3.4.2
(g) Theorem 4.2.2
(h) Theorem 5.1.7(a)

5. Which of the following conditional sentences are true?
★ (a) If triangles have three sides, then squares have four sides.
(b) If a hexagon has six sides, then the moon is made of cheese.
★ (c) If $7 + 6 = 14$, then $5 + 5 = 10$.
(d) If $5 < 2$, then $10 < 7$.
★ (e) If one interior angle of a right triangle is 92°, then the other interior angle is 88°.
(f) If Euclid's birthday was April 2, then rectangles have four sides.
(g) 5 is prime if $\sqrt{2}$ is not irrational.
(h) $1 + 1 = 2$ is sufficient for $3 > 6$.

6. Which of the following are true?
★ (a) Triangles have three sides iff squares have four sides.
(b) $7 + 5 = 12$ iff $1 + 1 = 2$.
★ (c) b is even iff $b + 1$ is odd. (Assume that b is some fixed integer.)
(d) m is odd iff m^2 is odd. (Assume that m is some fixed integer.)
(e) $5 + 6 = 6 + 5$ iff $7 + 1 = 10$.
(f) A parallelogram has three sides iff 27 is prime.
(g) The Eiffel Tower is in Paris if and only if the chemical symbol for helium is H.
(h) $\sqrt{10} + \sqrt{13} < \sqrt{11} + \sqrt{12}$ iff $\sqrt{13} - \sqrt{12} < \sqrt{11} - \sqrt{10}$.
(i) $x^2 \geq 0$ iff $x \geq 0$. (Assume that x is a fixed real number.)
(j) $x^2 - y^2 = 0$ iff $(x - y)(x + y) = 0$. (Assume that x and y are fixed real numbers.)
(k) $x^2 + y^2 = 50$ iff $(x + y)^2 = 50$. (Assume that x and y are fixed real numbers.)

7. Make truth tables for these propositional forms.
(a) $P \Rightarrow (Q \wedge P)$.
★ (b) $(\sim P \Rightarrow Q) \vee (Q \Leftrightarrow P)$.
★ (c) $\sim Q \Rightarrow (Q \Leftrightarrow P)$.
(d) $(P \vee Q) \Rightarrow (P \wedge Q)$.
(e) $(P \wedge Q) \vee (Q \wedge R) \Rightarrow P \vee R$.
(f) $[(Q \Rightarrow S) \wedge (Q \Rightarrow R)] \Rightarrow [(P \vee Q) \Rightarrow (S \vee R)]$.

8. Prove Theorem 1.2.2 by constructing truth tables for each equivalence.

9. Determine whether each statement qualifies as a definition.

(a) $y = f(x)$ is a linear function when its graph is a straight line.
(b) $y = f(x)$ is a quadratic function when it contains an x^2 term.
(c) m is a perfect square when $m = n^2$ for some integer n.
(d) A triangle is a right triangle when the sum of two of its interior angles is 90°.
(e) Two lines are parallel when their slopes are the same number.
(f) A sundial is an instrument for measuring time.

10. Rewrite each of the following sentences using logical connectives. Assume that each symbol f, x_0, n, x, S, $\mathbf{B}$ represents some fixed object.

★ **(a)** If f has a relative minimum at x_0 and if f is differentiable at x_0, then $f'(x_0) = 0$.
(b) If n is prime, then $n = 2$ or n is odd.
(c) A number x is real and not rational whenever x is irrational.
★ **(d)** If $x = 1$ or $x = -1$, then $|x| = 1$.
★ **(e)** f has a critical point at x_0 iff $f'(x_0) = 0$ or $f'(x_0)$ does not exist.
(f) S is compact iff S is closed and bounded.
(g) $\mathbf{B}$ is invertible is a necessary and sufficient condition for det $\mathbf{B} \neq 0$.
(h) $6 \geq n - 3$ only if $n > 4$ or $n > 10$.
(i) x is Cauchy implies x is convergent.
(j) f is continuous at x_0 whenever $\lim_{x \to x_0} f(x) = f(x_0)$.
(k) If f is differentiable at x_0 and f is strictly increasing at x_0, then $f'(x_0) > 0$.

11. Dictionaries indicate that the conditional meaning of *unless* is preferred, but there are other interpretations as a converse or a biconditional. Discuss the translation of each sentence.

(a) I will go to the store unless it is raining.
★ **(b)** The Dolphins will not make the playoffs unless the Bears win all the rest of their games.
(c) You cannot go to the game unless you do your homework first.
(d) You won't win the lottery unless you buy a ticket.

12. Show that the following pairs of statements are equivalent.

(a) $(P \vee Q) \Rightarrow R$ and $\sim R \Rightarrow (\sim P \wedge \sim Q)$.
★ **(b)** $(P \wedge Q) \Rightarrow R$ and $(P \wedge \sim R) \Rightarrow \sim Q$.
(c) $P \Rightarrow (Q \wedge R)$ and $(\sim Q \vee \sim R) \Rightarrow \sim P$.
(d) $P \Rightarrow (Q \vee R)$ and $(P \wedge \sim R) \Rightarrow Q$.
(e) $(P \Rightarrow Q) \Rightarrow R$ and $(P \wedge \sim Q) \vee R$.
(f) $P \Leftrightarrow Q$ and $(\sim P \vee Q) \wedge (\sim Q \vee P)$.

13. Give, if possible, an example of a true conditional sentence for which

★ **(a)** the converse is true. **(b)** the converse is false.
★ **(c)** the contrapositive is false. **(d)** the contrapositive is true.

14. Give, if possible, an example of a false conditional sentence for which

(a) the converse is true. **(b)** the converse is false.
(c) the contrapositive is false. **(d)** the contrapositive is true.

15. Give the converse and contrapositive of each sentence of Exercises 10(a), (b), (c), and (d). Tell whether each converse and contrapositive is true or false.

16. Determine whether each of the following is a tautology, a contradiction, or neither.

★ **(a)** $[(P \Rightarrow Q) \Rightarrow P] \Rightarrow P$.
(b) $P \Leftrightarrow P \wedge (P \vee Q)$.
(c) $P \Rightarrow Q \Leftrightarrow P \wedge \sim Q$.
★ **(d)** $P \Rightarrow [P \Rightarrow (P \Rightarrow Q)]$.
(e) $P \wedge (Q \vee \sim Q) \Leftrightarrow P$.
(f) $[Q \wedge (P \Rightarrow Q)] \Rightarrow P$.
(g) $(P \Leftrightarrow Q) \Leftrightarrow \sim(\sim P \vee Q) \vee (\sim P \wedge Q)$.
(h) $[P \Rightarrow (Q \vee R)] \Rightarrow [(Q \Rightarrow R) \vee (R \Rightarrow P)]$.
(i) $P \wedge (P \Leftrightarrow Q) \wedge \sim Q$.
(j) $(P \vee Q) \Rightarrow Q \Rightarrow P$.
(k) $[P \Rightarrow (Q \wedge R)] \Rightarrow [R \Rightarrow (P \Rightarrow Q)]$.
(l) $[P \Rightarrow (Q \wedge R)] \Rightarrow R \Rightarrow (P \Rightarrow Q)$.

17. The **inverse,** or **opposite,** of the conditional sentence $P \Rightarrow Q$ is $\sim P \Rightarrow \sim Q$.

(a) Show that $P \Rightarrow Q$ and its inverse are not equivalent forms.
(b) For what values of the propositions P and Q are $P \Rightarrow Q$ and its inverse both true?
(c) Which is equivalent to the converse of a conditional sentence, the contrapositive of its inverse, or the inverse of its contrapositive?

1.3 Quantifiers

Unless there has been a prior agreement about the value of x, the statement "$x \geq 3$" is neither true nor false. A sentence that contains variables is called an **open sentence** or **predicate**, and becomes a proposition only when its variables are assigned specific values. For example, "$x \geq 3$" is true when x is given the value 7 and false when $x = 2$.

When P is an open sentence with a variable x, the sentence is symbolized by $P(x)$. Likewise, if P has variables $x_1, x_2, x_3, \ldots, x_n$, the sentence may be denoted by $P(x_1, x_2, x_3, \ldots, x_n)$. For example, for the sentence "$x + y = 3z$" we write $P(x, y, z)$, and we see that $P(4, 5, 3)$ is true because $4 + 5 = 3(3)$, while $P(1, 2, 4)$ is false.

The collection of objects that may be substituted to make an open sentence a true proposition is called the **truth set** of the sentence. Before a truth set can be determined, we must be given or must decide what objects are available for consideration; that is, we must have specified a **universe of discourse.** In many cases the universe will be understood from the context. For a sentence such as "x likes chocolate," the universe is presumably the set of all people. We will often use the number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ as our universes. (See the *Preface to the Student*.)

Example. The truth set of the open sentence "$x^2 < 5$" depends upon the collection of objects we choose for the universe of discourse. With the universe specified as the set $\mathbb{N}$, the truth set is $\{1, 2\}$. For the universe $\mathbb{Z}$, the truth set is $\{-2, -1, 0, 1, 2\}$. When the universe is $\mathbb{R}$, the truth set is the open interval $(-\sqrt{5}, \sqrt{5})$.