15. Give the converse and contrapositive of each sentence of Exercises 10(a), (b), (c), and (d). Tell whether each converse and contrapositive is true or false.

16. Determine whether each of the following is a tautology, a contradiction, or neither.

★ **(a)** $[(P \Rightarrow Q) \Rightarrow P] \Rightarrow P$.

(b) $P \Leftrightarrow P \wedge (P \vee Q)$.

(c) $P \Rightarrow Q \Leftrightarrow P \wedge \sim Q$.

★ **(d)** $P \Rightarrow [P \Rightarrow (P \Rightarrow Q)]$.

(e) $P \wedge (Q \vee \sim Q) \Leftrightarrow P$.

(f) $[Q \wedge (P \Rightarrow Q)] \Rightarrow P$.

(g) $(P \Leftrightarrow Q) \Leftrightarrow \sim(\sim P \vee Q) \vee (\sim P \wedge Q)$.

(h) $[P \Rightarrow (Q \vee R)] \Rightarrow [(Q \Rightarrow R) \vee (R \Rightarrow P)]$.

(i) $P \wedge (P \Leftrightarrow Q) \wedge \sim Q$.

(j) $(P \vee Q) \Rightarrow Q \Rightarrow P$.

(k) $[P \Rightarrow (Q \wedge R)] \Rightarrow [R \Rightarrow (P \Rightarrow Q)]$.

(l) $[P \Rightarrow (Q \wedge R)] \Rightarrow R \Rightarrow (P \Rightarrow Q)$.

17. The **inverse,** or **opposite,** of the conditional sentence $P \Rightarrow Q$ is $\sim P \Rightarrow \sim Q$.

(a) Show that $P \Rightarrow Q$ and its inverse are not equivalent forms.

(b) For what values of the propositions P and Q are $P \Rightarrow Q$ and its inverse both true?

(c) Which is equivalent to the converse of a conditional sentence, the contrapositive of its inverse, or the inverse of its contrapositive?

1.3 Quantifiers

Unless there has been a prior agreement about the value of x, the statement "$x \geq 3$" is neither true nor false. A sentence that contains variables is called an **open sentence** or **predicate**, and becomes a proposition only when its variables are assigned specific values. For example, "$x \geq 3$" is true when x is given the value 7 and false when $x = 2$.

When P is an open sentence with a variable x, the sentence is symbolized by $P(x)$. Likewise, if P has variables $x_1, x_2, x_3, \ldots, x_n$, the sentence may be denoted by $P(x_1, x_2, x_3, \ldots, x_n)$. For example, for the sentence "$x + y = 3z$" we write $P(x, y, z)$, and we see that $P(4, 5, 3)$ is true because $4 + 5 = 3(3)$, while $P(1, 2, 4)$ is false.

The collection of objects that may be substituted to make an open sentence a true proposition is called the **truth set** of the sentence. Before a truth set can be determined, we must be given or must decide what objects are available for consideration; that is, we must have specified a **universe of discourse.** In many cases the universe will be understood from the context. For a sentence such as "x likes chocolate," the universe is presumably the set of all people. We will often use the number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ as our universes. (See the *Preface to the Student*.)

Example. The truth set of the open sentence "$x^2 < 5$" depends upon the collection of objects we choose for the universe of discourse. With the universe specified as the set $\mathbb{N}$, the truth set is $\{1, 2\}$. For the universe $\mathbb{Z}$, the truth set is $\{-2, -1, 0, 1, 2\}$. When the universe is $\mathbb{R}$, the truth set is the open interval $(-\sqrt{5}, \sqrt{5})$.

DEFINITION With a universe specified, two open sentences $P(x)$ and $Q(x)$ are **equivalent** iff they have the same truth set.

Examples. The sentences "$3x + 2 = 20$" and "$x = 6$" are equivalent open sentences in any of the number systems we have named. On the other hand, "$x^2 = 4$" and "$x = 2$" are *not* equivalent when the universe is $\mathbb{R}$. They *are* equivalent when the universe is $\mathbb{N}$.

The notions of truth set, universe, and equivalent open sentences should not be new concepts for you. Solving an equation such as $(x^2 + 1)(x - 3) = 0$ is a matter of determining what objects x make the open sentence "$(x^2 + 1)(x - 3) = 0$" true. For the universe $\mathbb{R}$, the only solution is $x = 3$ and thus the truth set is $\{3\}$. But if we choose the universe to be $\mathbb{C}$, the equation may be replaced by the equivalent open sentence $(x + i)(x - i)(x - 3) = 0$, which has truth set (solutions) $\{3, i, -i\}$.

A sentence such as

"There is a prime number between 5060 and 5090"

is treated differently from the propositions we considered earlier. To determine whether this sentence is true in the universe $\mathbb{N}$, we might try to individually examine every natural number, checking whether it is a prime and between 5060 and 5090, until we eventually find any *one* of the primes 5077, 5081, or 5087 and conclude that the sentence is true. (A quicker way is to search through a complete list of the first thousand primes.) The key idea here is that although the open sentence "x is a prime number between 5060 and 5090" is not a proposition, the sentence

"There is a number x such that x is a prime number between 5060 and 5090"

does have a truth value. This sentence is formed from the original open sentence by applying a quantifier.

DEFINITION For an open sentence $P(x)$, the sentence $(\exists x)P(x)$ is read "There exists x such that $P(x)$" or "For some x, $P(x)$." The sentence $(\exists x)P(x)$ is true iff the truth set of $P(x)$ is nonempty. The symbol $\exists$ is called the **existential quantifier**.

An open sentence $P(x)$ does not have a truth value, but the quantified sentence $(\exists x)P(x)$ does. One way to show that $(\exists x)P(x)$ is true for a particular universe is to identify an object a in the universe such that the proposition $P(a)$ is true. To show $(\exists x)P(x)$ is false, we must show that the truth set of $P(x)$ is empty.

Examples. Let's examine the truth values of these statements for the universe $\mathbb{R}$:

(a) $(\exists x)(x \geq 3)$
(b) $(\exists x)(x^2 = 0)$
(c) $(\exists x)(x^2 = -1)$

Statement (a) is true because the truth set of $x \geq 3$ contains 3, 7.02, and many other real numbers. Thus the truth set contains at least one real number. Statement (b) is true because the truth set of $x^2 = 0$ is precisely $\{0\}$ and thus is nonempty. Since the open sentence $x^2 = -1$ is never true for real numbers, the truth set of $x^2 = -1$ is empty. Statement (c) is false.

In the universe $\mathbb{N}$, only statement (a) is true. The three statements are all true in the universe $\{0, 5, i\}$ and all three statements are false in the universe $\{1, 2\}$.

Sometimes we can say $(\exists x)P(x)$ is true even when we do not know a specific object in the universe in the truth set of $P(x)$, only that there (at least) is one.

Example. Show that $(\exists x)(x^7 - 12x^3 + 16x - 3 = 0)$ is true in the universe of real numbers.

For the polynomial $f(x) = x^7 - 12x^3 + 16x - 3$, $f(0) = -3$ and $f(1) = 2$. From calculus, we know that f is continuous on $[0, 1]$. The Intermediate Value Theorem tells us there is a zero for f between 0 and 1. Even if we don't know the exact value of the zero, we know it exists. Therefore, the truth set of $x^7 - 12x^3 + 16x - 3 = 0$ is nonempty. Hence $(\exists x)(x^7 - 12x^3 + 16x - 3 = 0)$ is true.

The sentence "The square of every number is greater than 3" uses a different quantifier for the open sentence "$x^2 > 3$." To decide the truth value of the given sentence in the universe $\mathbb{N}$ it is not enough to observe that $3^2 > 3$, $4^2 > 3$, and so on. In fact, the sentence is false in $\mathbb{N}$ because 1 is in the universe but not in the truth set. The sentence is true, however, in the universe $[1.74, \infty)$ because with this universe the truth set for $x^2 > 3$ is the same as the universe.

DEFINITION For an open sentence $P(x)$, the sentence $(\forall x)P(x)$ is read "For all x, $P(x)$" and is true iff the truth set of $P(x)$ is the *entire* universe. The symbol $\forall$ is called the **universal quantifier**.

Examples. For the universe of all real numbers,

$(\forall x)(x + 2 > x)$ is true.
$(\forall x)(x > 0 \vee x = 0 \vee x < 0)$ is true. That is, every real number is positive, zero or negative.
$(\forall x)(x \geq 3)$ is false because there are (many) real numbers x for which $x \geq 3$ is false.
$(\forall x)(|x| > 0)$ is false, because 0 is not in the truth set.

There are many ways to express a quantified sentence in English. Look for key words such as "for all," "for every," "for each," or similar words that require universal quantifiers. Look for phrases such as "some," "at least one," "there exist(s)," "there is (are)," and others that indicate existential quantifiers.

You should also be alert for hidden quantifiers because natural languages allow for imprecise quantified statements where the words "for all" and "there exists" are not present. Someone who says "Polynomial functions are continuous" means that "All polynomial functions are continuous," but someone who says "Rational functions have vertical asymptotes" must mean "Some rational functions have vertical asymptotes."

We agree that "All apples have spots" is quantified with $\forall$, but what form does it have? If we limit the universe to just apples, a correct symbolization would be $(\forall x)(x$ has spots). But if the universe is all fruits, we need to be more careful. Let $A(x)$ be "x is an apple" and $S(x)$ be "x has spots." Should we write the sentence as $(\forall x)[A(x) \wedge S(x)]$ or $(\forall x)[A(x) \Rightarrow S(x)]$?

The first quantified form, $(\forall x)[A(x) \wedge S(x)]$, says "For all objects x in the universe, x is an apple and x has spots." Since we don't really intend to say that all fruits are spotted apples, this is not the meaning we want. Our other choice, $(\forall x)[A(x) \Rightarrow S(x)]$, is the correct one because it says "For all objects x in the universe, if x is an apple then x has spots." In other words, "If a fruit is an apple, then it has spots."

Now consider "Some apples have spots." Should this be $(\exists x)[A(x) \wedge S(x)]$ or $(\exists x)[A(x) \Rightarrow S(x)]$? The first form says "There is an object x such that it is an apple and it has spots," which is correct. On the other hand, $(\exists x)[A(x) \Rightarrow S(x)]$ reads "There is an object x such that, if it is an apple, then it has spots," which does *not* ensure the existence of apples with spots. The sentence $(\exists x)[A(x) \Rightarrow S(x)]$ is true in every universe for which there is an object x such that either x is not an apple or x has spots, which is not the meaning we want.

In general, a sentence of the form "All $P(x)$ are $Q(x)$" should be symbolized $(\forall x)[P(x) \Rightarrow Q(x)]$. And, *in general*, a sentence of the form "Some $P(x)$ are $Q(x)$" should be symbolized $(\exists x)[P(x) \wedge Q(x)]$.

Examples. The sentence "For every odd prime x less than 10, $x^2 + 4$ is prime" means that if x is prime, and odd, and less than 10, then $x^2 + 4$ is prime. It is written symbolically as

$$(\forall x)(x \text{ is prime } \wedge\ x \text{ is odd} \wedge x < 10 \Rightarrow x^2 + 4 \text{ is prime}).$$

The sentence "Some functions defined at 0 are not continuous at 0" can be written symbolically as $(\exists f)(f$ is defined at $0 \wedge f$ is not continuous at 0).

Example. The sentence "Some real numbers have a multiplicative inverse" could be symbolized

$$(\exists x)(x \text{ is a real number} \wedge x \text{ has a real multiplicative inverse}).$$

However, "x has an inverse" means there is some number that is an inverse for x (hidden quantifier), so a more complete symbolic translation is

$$(\exists x)[x \text{ is a real number} \wedge (\exists y)(y \text{ is a real number} \wedge xy = 1)].$$

Example. One correct translation of "Some integers are even and some integers are odd" is

$$(\exists x)(x \text{ is even}) \wedge (\exists x)(x \text{ is odd})$$

because the first quantifier $(\exists x)$ extends only as far as the word "even." After that, any variable (even x again) may be used to express "some are odd." It would be equally correct and sometimes preferable to write

$$(\exists x)(x \text{ is even}) \wedge (\exists y)(y \text{ is odd}),$$

but it would be wrong to write

$$(\exists x)(x \text{ is even} \wedge x \text{ is odd}),$$

because there is no integer that is both even and odd.

Several of our essential definitions given in the *Preface to the Student* are in fact quantified statements. For example, the definition of a rational number may be symbolized:

$$r \text{ is a rational number iff } (\exists p)(\exists q)\left(p \in \mathbb{Z} \wedge q \in \mathbb{Z} \wedge q \neq 0 \wedge r = \tfrac{p}{q}\right)$$

Statements of the form "Every element of the set A has the property P" and "Some element of the set A has property P" occur so frequently that abbreviated symbolic forms are desirable. "Every element of the set A has the property P" could be restated as "If $x \in A$, then . . ." and symbolized by

$$(\forall x \in A)\, P(x).$$

"Some element of the set A has property P" is abbreviated by

$$(\exists x \in A)\, P(x).$$

Examples. The definition of a rational number given above may be written as

$$r \text{ is a rational number iff } (\exists p \in \mathbb{Z})(\exists q \in \mathbb{Z})\left(q \neq 0 \wedge r = \tfrac{p}{q}\right).$$

The statement "For every rational number there is a larger integer" may be symbolized by

$$(\forall x)[x \in \mathbb{Q} \Rightarrow (\exists z)(z \in \mathbb{Z} \text{ and } z > x)]$$

or

$$(\forall x \in \mathbb{Q})(\exists z \in \mathbb{Z})(z > x).$$

DEFINITION Two quantified sentences are **equivalent in a given universe** iff they have the same truth value in that universe. Two quantified sentences are **equivalent** iff they are equivalent in every universe.

Example. $(\forall x)(x > 3)$ and $(\forall x)(x \geq 4)$ are equivalent in the universe of integers (because both are false), the universe of natural numbers greater than 10 (because both are true), and in many other universes. However, if we chose a number between 3 and 4, say 3.7, and let U be the universe of real numbers larger than 3.7, then $(\forall x)(x > 3)$ is true and $(\forall x)(x \geq 4)$ is false in U. The sentences are not equivalent in this universe, so they are not equivalent sentences.

As was noted with propositional forms, it is necessary to make a distinction between a quantified sentence and its logical form. With the universe all integers, the sentence "All integers are odd" is an instance of the logical form $(\forall x)P(x)$, where $P(x)$ is "x is odd." The form itself, $(\forall x)P(x)$, is neither true nor false, but becomes false when "x is odd" is substituted for $P(x)$ and the universe is all integers.

The pair of quantified forms $(\exists x)([P(x) \wedge Q(x)]$ and $(\exists x)([Q(x) \wedge P(x)]$ are equivalent because for any choices of P and Q, $P \wedge Q$ and $Q \wedge P$ are equivalent propositional forms. Another pair of equivalent sentences is $(\forall x)[P(x) \Rightarrow Q(x)]$ and $(\forall x)[\sim Q(x) \Rightarrow \sim P(x)]$.

The next two equivalences are essential for reasoning about quantifiers.

Theorem 1.3.1 If $A(x)$ is an open sentence with variable x, then

(a) $\sim(\forall x)A(x)$ is equivalent to $(\exists x) \sim A(x)$.
(b) $\sim(\exists x)A(x)$ is equivalent to $(\forall x) \sim A(x)$.

Proof.
(a) Let U be any universe.
The sentence $\sim(\forall x)A(x)$ is true in U
iff $(\forall x)A(x)$ is false in U
iff the truth set of $A(x)$ is not the universe
iff the truth set of $\sim A(x)$ is nonempty
iff $(\exists x) \sim A(x)$ is true in U.

(b) The proof of this part is Exercise 7. ■

Theorem 1.3.1 is helpful for finding useful denials (that is, simplified forms of negations) of quantified sentences. For example, in the universe of natural numbers, the sentence "All primes are odd" is symbolized $(\forall x)(x \text{ is prime} \Rightarrow x \text{ is odd})$. The negation is $\sim(\forall x)(x \text{ is prime} \Rightarrow x \text{ is odd})$. By applying Theorem 1.3.1(a), this becomes $(\exists x)[\sim(x \text{ is prime} \Rightarrow x \text{ is odd})]$. By Theorem 1.2.2(c) this is equivalent to $(\exists x)[x \text{ is prime} \wedge \sim(x \text{ is odd})]$. We read this last statement as "There exists a number that is prime and is not odd" or "Some prime number is even."

Example. A simplified denial of $(\forall x)(\exists y)(\exists z)(\forall u)(\exists v)(x + y + z > 2u + v)$ begins with its negation

$$\sim(\forall x)(\exists y)(\exists z)(\forall u)(\exists v)(x + y + z > 2u + v).$$

After 5 applications of Theorem 1.3.1, beginning with the outermost quantifier $(\forall x)$, we arrive at the simplified form

$$(\exists x)(\forall y)(\forall z)(\exists u)(\forall v)(x + y + z \leq 2u + v).$$

Example. For the universe of all real numbers, find a denial of "Every positive real number has a multiplicative inverse."

The sentence is symbolized $(\forall x)[x > 0 \Rightarrow (\exists y)(xy = 1)]$. The negation and successively rewritten equivalents are:

$$\sim(\forall x)[x > 0 \Rightarrow (\exists y)(xy = 1)]$$
$$(\exists x) \sim[x > 0 \Rightarrow (\exists y)(xy = 1)]$$
$$(\exists x)[x > 0 \wedge \sim(\exists y)(xy = 1)]$$
$$(\exists x)[x > 0 \wedge (\forall y) \sim(xy = 1)]$$
$$(\exists x)[x > 0 \wedge (\forall y)(xy \neq 1)]$$

This last sentence may be translated as "There is a positive real number that has no multiplicative inverse."

Example. For the universe of living things, find a denial of "Some children do not like clowns."

The sentence is $(\exists x)$ [x is a child $\wedge$ $(\forall y)$(y is a clown $\Rightarrow$ x does not like y)]. Its negation and several equivalents are:

$\sim(\exists x)$ [x is a child $\wedge$ $(\forall y)$(y is a clown $\Rightarrow$ x does not like y)]

$(\forall x) \sim$[x is a child $\wedge$ $(\forall y)$(y is a clown $\Rightarrow$ x does not like y)]

$(\forall x)$[x is a child $\Rightarrow$ $\sim(\forall y)$(y is a clown $\Rightarrow$ x does not like y)]

$(\forall x)$[x is a child $\Rightarrow$ $(\exists y) \sim$(y is a clown $\Rightarrow$ x does not like y)]

$(\forall x)$[x is a child $\Rightarrow$ $(\exists y)$(y is a clown $\wedge \sim x$ does not like y)]

$(\forall x)$[x is a child $\Rightarrow$ $(\exists y)$(y is a clown $\wedge$ x likes y)]

The denial we seek is "Every child has some clown that he/she likes."

We sometimes hear statements like the complaint one fan had after a great Little League baseball game. "The game was fine," he said, "but everybody didn't get to play." We easily understand that the fan did not mean this literally, because otherwise there would have been no game. The meaning we understand is "Not everyone got to play" or "Some team members did not play." Such misuse of quantifiers, while tolerated in casual conversations, is always to be avoided in mathematics.

The $\exists!$ quantifier, defined next, is a special case of the existential quantifier.

DEFINITION For an open sentence $P(x)$, the proposition $(\exists! x) P(x)$ is read "**there exists a unique x such that $P(x)$**" and is true iff the truth set of $P(x)$ has *exactly one element*. The symbol $\exists!$ is called the **unique existential quantifier.**

Recall that for $(\exists x)P(x)$ to be true it is unimportant how many elements are in the truth set of $P(x)$, as long as there is at least one. For $(\exists!x)P(x)$ to be true, the number of elements in the truth set of $P(x)$ is crucial—there must be exactly one.

In the universe of natural numbers, $(\exists!x)$ (x is even and x is prime) is true because the truth set of "x is even and x is prime" contains only the number 2. The sentence $(\exists!x)(x^2 = 4)$ is true in the universe of natural numbers, but false in the universe of all integers.

Theorem 1.3.2 If $A(x)$ is an open sentence with variable x, then

(a) $(\exists!x)A(x) \Rightarrow (\exists x)A(x)$.
(b) $(\exists!x)A(x)$ is equivalent to $(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \Rightarrow y = z)$.

Part (a) of Theorem 1.3.2 says that $\exists!$ is indeed a special case of the quantifier $\exists$. Part (b) says that "There exists a unique x such that $A(x)$" is equivalent to "There is an x such that $A(x)$ and if both $A(y)$ and $A(z)$, then $y = z$." The proofs are left to Exercise 11.

Exercises 1.3

1. Translate the following English sentences into symbolic sentences with quantifiers. The universe for each is given in parentheses.
   ★ **(a)** Not all precious stones are beautiful. (All stones)
   ☆ **(b)** All precious stones are not beautiful. (All stones)
   (c) Some isosceles triangle is a right triangle. (All triangles)
   (d) No right triangle is isosceles. (All triangles)
   (e) All people are honest or no one is honest. (All people)
   (f) Some people are honest and some people are not honest. (All people)
   (g) Every nonzero real number is positive or negative. (Real numbers)
   ★ **(h)** Every integer is greater than -4 or less than 6. (Real numbers)
   (i) Every integer is greater than some integer. (Integers)
   ★ **(j)** No integer is greater than every other integer. (Integers)
   (k) Between any integer and any larger integer, there is a real number. (Real numbers)
   ★ **(l)** There is a smallest positive integer. (Real numbers)
   ★ **(m)** No one loves everybody. (All people)
   (n) Everybody loves someone. (All people)
   (o) For every positive real number x, there is a unique real number y such that $2^y = x$. (Real numbers)

☆ 2. For each of the propositions in Exercise 1, write a useful denial, and give a translation into ordinary English.

3. Translate these definitions from the *Preface to the Student* into quantified sentences.
   (a) The integer x is **even**.
   (b) The integer x is **odd**.

(c) The integer a **divides** the integer b.
(d) The natural number n is **prime**.
(e) The natural number n is **composite**.

4. Translate these definitions in this text into quantified sentences. You need not know the specifics of the terms and symbols to complete this exercise.
   (a) The relation R is **symmetric**. (See page 147.)
   (b) The relation R is **transitive**. (See page 147.)
   (c) The function f is **one-to-one**. (See page 208.)
   (d) The operation $*$ is **commutative**. (See page 277.)

☆ 5. The sentence "People dislike taxes" might be interpreted to mean "All people dislike all taxes," "All people dislike some taxes," "Some people dislike all taxes," or "Some people dislike some taxes." Give a symbolic translation for each of these interpretations.

6. Let $T = \{17\}$, $U = \{6\}$, $V = \{24\}$, and $W = \{2, 3, 7, 26\}$. In which of these four different universes is the statement true?
   ★ a) $(\exists x)(x \text{ is odd} \Rightarrow x > 8)$.
   b) $(\exists x)(x \text{ is odd} \wedge x > 8)$.
   c) $(\forall x)(x \text{ is odd} \Rightarrow x > 8)$.
   d) $(\forall x)(x \text{ is odd} \wedge x > 8)$.

7. (a) Complete this proof of Theorem 1.3.1(b):
   Proof: Let U be any universe.
   The sentence $\sim(\exists x)A(x)$ is true in U
   iff . . .
   iff $(\forall x) \sim A(x)$ is true in U.
   ☆ (b) Give a proof of part (b) of Theorem 1.3.1 that uses part (a).

8. Which of the following are true? The universe for each statement is given in parentheses.
   (a) $(\forall x)(x + x \geq x)$. $(\mathbb{R})$
   ★ (b) $(\forall x)(x + x \geq x)$. $(\mathbb{N})$
   (c) $(\exists x)(2x + 3 = 6x + 7)$. $(\mathbb{N})$
   (d) $(\exists x)(3^x = x^2)$. $(\mathbb{R})$
   ★ (e) $(\exists x)(3^x = x)$. $(\mathbb{R})$
   (f) $(\exists x)(3(2 - x) = 5 + 8(1 - x))$. $(\mathbb{R})$
   (g) $(\forall x)(x^2 + 6x + 5 \geq 0)$. $(\mathbb{R})$
   ★ (h) $(\forall x)(x^2 + 4x + 5 \geq 0)$. $(\mathbb{R})$
   (i) $(\exists x)(x^2 - x + 41 \text{ is prime})$. $(\mathbb{N})$
   (j) $(\forall x)(x^2 - x + 41 \text{ is prime})$. $(\mathbb{N})$
   (k) $(\forall x)(x^3 + 17x^2 + 6x + 100 \geq 0)$. $(\mathbb{R})$
   (l) $(\forall x)(\forall y)[x < y \Rightarrow (\exists w)(x < w < y)]$. $(\mathbb{Q})$

9. Give an English translation for each. The universe is given in parentheses.
   (a) $(\forall x)(x \geq 1)$. $(\mathbb{N})$
   ★ (b) $(\exists! x)(x \geq 0 \wedge x \leq 0)$. $(\mathbb{R})$
   (c) $(\forall x)(x \text{ is prime} \wedge x \neq 2 \Rightarrow x \text{ is odd})$. $(\mathbb{N})$
   ★ (d) $(\exists! x)(\log_e x = 1)$. $(\mathbb{R})$

(e) $\sim(\exists x)(x^2 < 0)$. ($\mathbb{R}$)
(f) $(\exists! x)(x^2 = 0)$. ($\mathbb{R}$)
(g) $(\forall x)(x \text{ is odd} \Rightarrow x^2 \text{ is odd})$. ($\mathbb{N}$)

10. Which of the following are true in the universe of all real numbers?
★ **(a)** $(\forall x)(\exists y)(x + y = 0)$.
(b) $(\exists x)(\forall y)(x + y = 0)$.
(c) $(\exists x)(\exists y)(x^2 + y^2 = -1)$.
★ **(d)** $(\forall x)[x > 0 \Rightarrow (\exists y)(y < 0 \wedge xy > 0)]$.
(e) $(\forall y)(\exists x)(\forall z)(xy = xz)$.
★ **(f)** $(\exists x)(\forall y)(x \leq y)$.
(g) $(\forall y)(\exists x)(x \leq y)$.
(h) $(\exists! y)(y < 0 \wedge y + 3 > 0)$.
★ **(i)** $(\exists! x)(\forall y)(x = y^2)$.
(j) $(\forall y)(\exists! x)(x = y^2)$.
(k) $(\exists! x)(\exists! y)(\forall w)(w^2 > x - y)$.

11. Let $A(x)$ be an open sentence with variable x.
☆ **(a)** Prove Theorem 1.3.2 (a).
☆ **(b)** Show that the converse of Theorem 1.3.2 (a) is false.
(c) Prove Theorem 1.3.2 (b).
(d) Prove that $(\exists! x)A(x)$ is equivalent to $(\exists x)[A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)]$.
★ **(e)** Find a useful denial for $(\exists! x)A(x)$.

12. **(a)** Write the symbolic form for the definition of "f is continuous at a."
(b) Write the symbolic form of the statement of the Mean Value Theorem.
(c) Write the symbolic form for the definition of "$\lim_{x \to a} f(x) = L$."
(d) Write a useful denial of each sentence in parts (a), (b), and (c).

13. Which of the following are denials of $(\exists! x)P(x)$?
(a) $(\forall x)P(x) \vee (\forall x)\sim P(x)$.
(b) $(\forall x)\sim P(x) \vee (\exists y)(\exists z)(y \neq z \wedge P(y) \wedge P(z))$.
(c) $(\forall x)[P(x) \Rightarrow (\exists y)(P(y) \wedge x \neq y)]$.
★ **(d)** $\sim(\forall x)(\forall y)[(P(x) \wedge P(y)) \Rightarrow x = y]$.

★ **14.** *Riddle:* What is the English translation of the symbolic statement $\forall\exists\exists\forall$?

1.4 Basic Proof Methods I

In mathematics, a **theorem** is a statement that describes a pattern or relationship among quantities or structures and a **proof** is a justification of the truth of a theorem. Before beginning to examine valid proof techniques it is recommended that you review the comments about proofs and the definitions in the *Preface to the Student.*

We cannot define all terms nor prove all statements from previous ones. We begin with an initial set of statements, called **axioms** (or **postulates**), that are *assumed to be true*. We then derive theorems that are true in any situation where the axioms are true. The Pythagorean* Theorem, for example, is a theorem whose proof is ultimately based on the five axioms of Euclidean† geometry. In a situation where the Euclidean axioms are not all true (which can happen), the Pythagorean Theorem may not be true.

There must also be an initial set of **undefined terms**—concepts fundamental to the context of study. In geometry, the concept of a point is an undefined term. In this text the real numbers are not formally defined. Instead, they are described in the *Preface to the Student* as the decimal numbers along the number line. While a precise definition of a real number could be given‡, doing so would take us far from our intended goals.

From the axioms and undefined terms, new concepts (new **definitions**) can be introduced. And finally, new theorems can be proved. The structure of a proof for a particular theorem depends greatly on the logical form of the theorem. Proofs may require some ingenuity or insightfulness to put together the right statements to build the justification. Nevertheless, much can be gained in the beginning by studying the fundamental components found in proofs and examples that exhibit them. The four rules that follow provide guidance about what statements are allowed in a proof, and when.

Some steps in a proof may be statements of axioms of the basic theory upon which the discussion rests. Other steps may be previously proved results. Still other steps may be assumptions you wish to introduce. In any proof you may

At any time state an assumption, an axiom, or a previously proved result.

The statement of an assumption generally takes the form "Assume P" to alert the reader that the statement is not derived from a previous step or steps. We must be careful about making assumptions, because we can only be certain that what we proved will be true *when all the assumptions are true*. The most common assumptions are hypotheses given as components in the statement of the theorem to be proved. We will discuss assumptions in more detail later in this section.

The statement of an axiom is usually easily identified as such by the reader because it is a statement about a very fundamental fact assumed about the theory. Sometimes the axiom is so well known that its statement is omitted from proofs, but there are cases (such as the Axiom of Choice in Chapter 5) for which it is prudent to mention the axiom in every proof employing it.

Proof steps that use previously proven results help build a rich theory from the basic assumptions. In calculus, for example, before one proves that the derivative of $\sin x$ is $\cos x$, there is a proof of the separate result that $\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1$. It is easier to prove this result first, then cite the result in the proof of the fact that the derivative of $\sin x$ is $\cos x$.

* Pythagoras, latter half of the 6th century, B.C.E., was a Greek mathematician and philosopher who founded a secretive religious society based on mathematical and metaphysical thought. Although Pythagoras is regularly given credit for the theorem named for him, the result was known to Babylonian and Indian mathematicians centuries earlier.

† Euclid of Alexandria, circa 300 B.C.E., made his immortal contribution to mathematics with his famous text on geometry and number theory. His *Elements* sets forth a small number of axioms from which additional definitions and many familiar geometric results were developed in a rigorous way. Other geometries, based on different sets of axioms, did not begin to appear until the 1800s.

‡ See the references cited in Section 7.5.

An important skill for proof writing is the ability to rewrite a complex statement in an equivalent form that is more useful or helps to clarify its meaning. You may:

At any time state a sentence equivalent to any statement earlier in the proof.

This **replacement rule** is often used in combination with the equivalences of Theorems 1.1.1 and 1.2.2 to rewrite a statement involving logical connectives. For example, suppose we have been able to establish the step

"It is not the case that x is even and prime."

Because the form of this statement is $\sim(P \wedge Q)$, where P is "x is even" and Q is "x is prime," we may deduce that

"x is not even or x is not prime,"

which has form $\sim P \vee \sim Q$. We have applied the replacement rule, using one of De Morgan's Laws. A working knowledge of the equivalences of Theorems 1.1.1 and 1.2.2 is essential.

The replacement rule allows you to use definitions in two ways. First, if you are told or have shown that x is odd, then you can correctly state that for some natural number k, $x = 2k + 1$. You now have an equation to use. Second, if you need to prove that x is odd, then the definition gives you something equivalent to work toward: It suffices to show that x can be expressed as $x = 2k + 1$, for some natural number k. You'll find it useful in writing proofs to keep in mind these two ways we use definitions.

Example. If a proof contains the line "The product of real numbers a and b is zero," we could assert that "Either $a = 0$ or $b = 0$." In this example, the equivalence of the two statements comes from our knowledge of the real numbers that $(ab = 0) \Leftrightarrow (a = 0 \text{ or } b = 0)$.

Tautologies are important both because a statement that has the form of a tautology may be used as a step in a proof, and because tautologies are used to create rules for making deductions in a proof. The **tautology rule** says that you may:

At any time state a sentence whose symbolic translation is a tautology.

For example, if a proof involves a real number x, you may at any time assert "Either $x > 0$ or $x \leq 0$," since this is an instance of the tautology $P \vee \sim P$.

The rules above allow us to reword a statement or say something that's always true or is assumed to be true. The next rule is the one that allows us to make a connection so that we can get from statement P to a *different* statement Q.

The most fundamental rule of reasoning is **modus ponens**, which is based on the tautology $[P \wedge (P \Rightarrow Q)] \Rightarrow Q$. As we have seen in Section 1.2, what this means is that when P and $P \Rightarrow Q$ are both true, we may deduce that Q must also be true. The **modus ponens rule** says you may:

At any time after P and $P \Rightarrow Q$ appear in a proof, state that Q is true.

Example. From calculus we know that if a function f is differentiable on an interval (a, b), then f is continuous on the interval (a, b). A proof writer who had already written:

f is differentiable on the interval (a, b)

could use modus ponens to write as a subsequent step:

Therefore f is continuous on the interval (a, b).

This deduction uses the statements D, $D \Rightarrow C$, and C, where D is the statement "f is differentiable on interval (a, b)" and C is "f is continuous on the interval (a, b)."

Notice that in this example it would make the proof shorter and easier to read if we didn't write out the sentence $D \Rightarrow C$ in the proof. This is because the connection between differentiability and continuity is a well-known theorem, which the proof writer may assume that the reader knows.

When we use modus ponens to deduce statement Q from P and $P \Rightarrow Q$, the statement P could be an instance of a tautology, a simple or compound proposition whose components are either hypotheses, axioms, earlier statements deduced in the proof, or statements of previously proved theorems. Likewise, $P \Rightarrow Q$ may have been deduced earlier in the proof or may be a previous theorem, axiom, or tautology.

Example. You are at a crime scene and have established the following facts:

(1) If the crime did not take place in the billiard room, then Colonel Mustard is guilty.

(2) The lead pipe is not the weapon.

(3) Either Colonel Mustard is not guilty or the weapon used was a lead pipe.

From these facts and modus ponens, you may construct a proof that shows the crime took place in the billiard room:

Proof.

Statement (1)	$\sim B \Rightarrow M$
Statement (2)	$\sim L$
Statement (3)	$\sim M \vee L$
Statements (1) and (2) and (3)	$(\sim B \Rightarrow M) \wedge \sim L \wedge (\sim M \vee L)$
Statements (1), (2), and (3) imply the crime took place in the billiard room.	$[(\sim B \Rightarrow M) \wedge \sim L \wedge (\sim M \vee L)] \Rightarrow B$ is a tautology (see Exercise 2).
Therefore, the crime took place in the billiard room.	B

■

The last three statements above are an application of the modus ponens rule: We deduced Q from the statements P and $P \Rightarrow Q$, where Q is B and P is $(\sim B \Rightarrow M) \wedge \sim L \wedge (\sim M \vee L)$.

The previous example shows the power of pure reasoning: It is the *forms* of the propositions and not their meanings that allowed us to make the deductions.

Because our proofs are always about mathematical phenomena, we also need to understand the subject matter of the proof—the concepts involved and how they are related. Therefore, when you develop a strategy to construct a proof, keep in mind both the logical form of the theorem's statement and the mathematical concepts involved.

You won't find truth tables displayed or referred to in proofs that you encounter in mathematics: It is expected that readers are familiar with the rules of logic and correct forms of proof. As a general rule, when you write a step in a proof, ask yourself if deducing that step is valid in the sense that it uses one of the four rules above. If the step follows as a result of the use of a tautology, it is not necessary to cite the tautology in your proof. In fact, with practice you should eventually come to write proofs without purposefully thinking about tautologies. What *is* necessary is that every step be justifiable.

The first—and most important—proof method is the **direct proof** of statement of the form $P \Rightarrow Q$, which proceeds in a step by step fashion from the antecedent P to the consequent Q. Since $P \Rightarrow Q$ is false only when P is true and Q is false, it suffices to show that this situation cannot happen. The direct way to proceed is to assume that P is true and show (deduce) that Q is also true. A direct proof of $P \Rightarrow Q$ will have the following form:

DIRECT PROOF OF $P \Rightarrow Q$

Proof.

Assume P.

$\vdots$

Therefore, Q.

Thus, $P \Rightarrow Q$. ■

Some of the examples that follow actually involve quantified sentences. Since we won't consider proofs with quantifiers until Section 1.6, you should imagine for now that a variable represents some fixed object. Out first example proves the familiar fact that "If x is odd, then $x + 1$ is even." You should think of x as being some particular integer.

Example. Let x be an integer. Prove that if x is odd, then $x + 1$ is even.

Proof. ⟨*The theorem has the form $P \Rightarrow Q$, where P is "x is odd" and Q is "$x + 1$ is even."*⟩ Let x be an integer. ⟨*We may assume this hypothesis since it is given in the statement of the theorem.*⟩ Suppose x is odd. ⟨*We assume that the antecedent P is true. The goal is to derive the consequent Q as our last step.*⟩ From the definition of odd, $x = 2k + 1$ for some integer k. ⟨*This deduction is the replacement* *of P by an equivalent statement—the definition of "odd." We now have an equation to use.*⟩ Then $x + 1 = (2k + 1) + 1$ for some integer k. ⟨*This is another replacement using an algebraic property of* $\mathbb{N}$.⟩ Since $(2k + 1) + 1 = 2k + 2 = 2(k + 1)$, $x + 1$ is the product of 2 and an integer. ⟨*Another equivalent using algebra.*⟩ Thus $x + 1$ is even. ⟨*We have deduced Q.*⟩

Therefore, if x is an odd integer, then $x + 1$ is even. ⟨*We conclude that* $P \Rightarrow Q$.⟩ ■

In this example, we did not worry about what would happen if x were not odd. Remember that it is appropriate to assume P is true when giving a direct proof of $P \Rightarrow Q$. (If P is false, it does not matter what the truth of Q is; the statement we are trying to prove, $P \Rightarrow Q$, will be true.) The process of assuming that the antecedent is true and proceeding step by step to show the consequent is true is what makes this type of proof direct.

This example also includes parenthetical comments offset by ⟨ . . . ⟩ and in italics to explain how and why a proof is proceeding as it is. Such comments are not a requisite part of the proof, but are inserted to help clarify the workings of the proof. The proof above would stand alone as correct with all the comments deleted, or it could be written in shorter form, as follows.

Proof. Let x be an integer. Suppose x is odd. Then $x = 2k + 1$ for some integer k. Then $x + 1 = (2k + 1) + 1 = 2k + 2 = 2(k + 1)$. Since $k + 1$ is an integer and $x + 1 = 2(k + 1)$, $x + 1$ is even.

Therefore, if x is an odd integer, then $x + 1$ is even. ■

Great latitude is allowed for differences in taste and style among proof writers. Generally, in advanced mathematics, only the minimum amount of explanation is included in a proof. The reader is expected to know the definitions and previous results and be able to fill in computations and deductions as necessary. In this text, we shall include parenthetical comments for more complete explanations.

Example. Suppose a, b, and c are integers. Prove that if a divides b and b divides c, then a divides c.

Proof. Let a, b, and c be integers. ⟨*We start by assuming that the hypothesis is true.*⟩ Suppose a divides b and b divides c. ⟨*The antecedent is the compound sentence "a divides b and b divides c."*⟩ Then $b = ak$ for some integer k and $c = bm$ for some integer m. ⟨*We replaced the assumptions by equivalents using the definition of "divides." Notice that we did not assume that k and m are the same integer.*⟩ ⟨*To show that a divides c, we must write c as a multiple of a.*⟩ Therefore, $c = bm = (ak)m = a(km)$. Then c is a multiple of a. ⟨*We use the fact that if k and m are integers, then km is an integer.*⟩

Therefore, if a divides b and b divides c, then a divides c. ■

Both of the above examples and many more to follow use the following strategy for developing a direct proof of a conditional sentence:

1. Determine precisely the hypotheses (if any) and the antecedent and consequent.
2. Replace (if necessary) the antecedent with a more usable equivalent.
3. Replace (if necessary) the consequent by something equivalent and more readily shown.
4. Beginning with the assumption of the antecedent, develop a chain of statements that leads to the consequent. Each statement in the chain must be deducible from its predecessors or other known results.

As you write a proof, be sure it is not just a string of symbols. Every step of your proof should express a complete sentence. Be sure to include important connective words.

Example. Suppose a, b, and c are integers. Prove that if a divides b and a divides c, then a divides $b - c$.

Proof. Suppose a, b, and c are integers and a divides b and a divides c. ⟨*Now use the definition of divides.*⟩ Then $b = an$ for some integer n and $c = am$ for some integer m. Thus, $b - c = an - am = a(n - m)$. Since $n - m$ is an integer ⟨*using the fact that the difference of two integers is an integer* ⟩, a divides $b - c$. ■

Our next example of a direct proof, which comes from an exercise in precalculus mathematics, involves a point (x, y) in the Cartesian plane (Figure 1.4.1). It uses algebraic properties available to students in such a class.

Example. Prove that if $x < -4$ and $y > 2$, then the distance from (x, y) to $(1, -2)$ is at least 6.

Proof. Assume that $x < -4$ and $y > 2$. Then $x - 1 < -5$, so $(x - 1)^2 > 25$. Also $y + 2 > 4$, so $(y + 2)^2 > 16$. Therefore,

$$\sqrt{(x-1)^2 + (y+2)^2} > \sqrt{25 + 16} > \sqrt{36},$$

so the distance from (x, y) to $(1, -2)$ is at least 6. ■

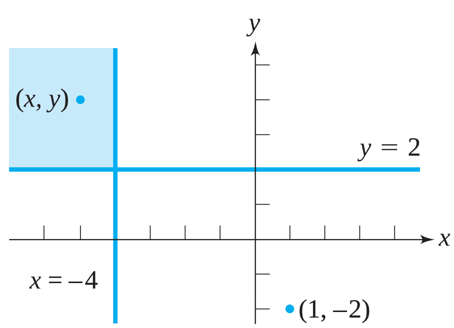

Figure 1.4.1

To get a sense of how a proof of $P \Rightarrow Q$ should proceed, it is sometimes useful to "work backward" from what is to be proved: To show that a consequent is true, decide what statement could be used to prove it, another statement that could be used to prove that one, and so forth. Continue until you reach a hypothesis, the antecedent, or a fact known to be true. After doing such preliminary work, construct a proof "forward" so that your conclusion is the consequent.

Example. Let a and b be positive real numbers. Prove that if $a < b$, then $b^2 - a^2 > 0$.

Proof. $\langle$*Working backward, rewrite* $b^2 - a^2 > 0$ *as* $(b - a)(b + a) > 0$. *This inequality will be true when both* $b - a > 0$ *and* $b + a > 0$. *The first inequality* $b - a > 0$ *will be true because we will assume the antecedent* $a < b$. *The second inequality* $b + a > 0$ *is true because of our hypothesis that a and b are positive. We now proceed with the direct proof.*$\rangle$ Assume a and b are positive real numbers and that $a < b$. Since both a and b are positive, $b + a > 0$. Since $a < b$, $b - a > 0$. Because the product of two positive real numbers is positive, $(b - a)(b + a) > 0$. Therefore $b^2 - a^2 > 0$. ∎

It is often helpful to work both ways—backward from what is to be proved and forward from the hypothesis—until you reach a common statement from each direction.

Example. Prove that if $x^2 \leq 1$, then $x^2 - 7x > -10$.

Working backward from $x^2 - 7x > -10$, we note that this can be deduced from $x^2 - 7x + 10 > 0$. This can be deduced from $(x - 5)(x - 2) > 0$, which could be concluded if we knew that $x - 5$ and $x - 2$ were both positive or both negative.

Working forward from $x^2 \leq 1$, we have $-1 \leq x \leq 1$, so $x \leq 1$. Therefore, $x < 5$ and $x < 2$, from which we can conclude that $x - 5 < 0$ and $x - 2 < 0$, which is exactly what we need.

Proof. Assume that $x^2 \leq 1$. Then $-1 \leq x \leq 1$. Therefore $x \leq 1$. Thus $x < 5$ and $x < 2$, and so we have $x - 5 < 0$ and $x - 2 < 0$. Therefore, $(x - 5)(x - 2) > 0$. Thus $x^2 - 7x + 10 > 0$. Hence $x^2 - 7x > -10$. ∎

We now consider direct proofs of statements of the form $P \Rightarrow Q$ when either P or Q is itself a compound proposition. We have in fact already constructed proofs of statements of the form $(P \wedge Q) \Rightarrow R$. When we give a direct proof of a statement of this form, we have the advantage of assuming both P and Q at the beginning of the proof, as we did in the proof (above) that if a divides b and a divides c, then a divides $b - c$.

A proof of a statement symbolized by $P \Rightarrow (Q \wedge R)$ would probably have two parts. In one part we prove $P \Rightarrow Q$ and in the other part we prove $P \Rightarrow R$. We would use this method to prove the statement "If two parallel lines are cut by a transversal, then corresponding angles are equal and corresponding lines are equal."

To prove a conditional sentence whose consequent is a disjunction, that is, a sentence of the form $P \Rightarrow (Q \vee R)$, one often proves either the equivalent $P \wedge {\sim}Q \Rightarrow R$ or the equivalent $P \wedge {\sim}R \Rightarrow Q$. For instance, to prove "If the polynomial *f* has degree 4, then *f* has a real zero or *f* can be written as the product of two irreducible quadratics," we would prove "If *f* has degree 4 and no real zeros, then *f* can be written as the product of two irreducible quadratics."

A statement of the form $(P \vee Q) \Rightarrow R$ has the meaning: "If either P is true or Q is true, then R is true," or "In case either P or Q is true, R must be true." A natural way to prove such a statement is by cases, so the proof outline would have the form:

Case 1. Assume P. . . . Therefore R.
Case 2. Assume Q. . . . Therefore R.

This method is valid because of the tautology

$$[(P \vee Q) \Rightarrow R] \Leftrightarrow [(P \Rightarrow R) \wedge (Q \Rightarrow R)].$$

The statement "If a quadrilateral has opposite sides equal or opposite angles equal, then it is a parallelogram" is proved by showing both "A quadrilateral with opposite sides equal is a parallelogram" and "A quadrilateral with opposite angles equal is a parallelogram."

The two similar statement forms $(P \Rightarrow Q) \Rightarrow R$ and $P \Rightarrow (Q \Rightarrow R)$ have remarkably dissimilar direct proof outlines. For $(P \Rightarrow Q) \Rightarrow R$, we assume $P \Rightarrow Q$ and deduce R. We cannot assume P; we must assume $P \Rightarrow Q$. On the other hand, in a direct proof of $P \Rightarrow (Q \Rightarrow R)$, we do assume P and show $Q \Rightarrow R$. Furthermore, after the assumption of P, a direct proof of $Q \Rightarrow R$ begins by assuming Q is true as well. This is not surprising since $P \Rightarrow (Q \Rightarrow R)$ is equivalent to $(P \wedge Q) \Rightarrow R$.

The main lesson to be learned from this discussion is that the method of proof you choose will depend on the form of the statement to be proved. The outlines we have given are the most natural, but not the only ways, to construct correct proofs. Of course constructing a proof also requires knowledge of the subject matter.

Example. Suppose n is an odd integer. Then $n = 4j + 1$ for some integer j, or $n = 4i - 1$ for some integer i.

Proof. Suppose n is odd. Then $n = 2m + 1$ for some integer m. ⟨*A little experimentation shows that when m is even, for example when n is* $2(-2) + 1$, $2(0) + 1$, $2(2) + 1$, $2(4) + 1$, *etc., n has the form* $4j + 1$; *otherwise n has the form* $4i - 1$. *We now show that* $(P \vee Q) \Rightarrow (R_1 \vee R_2)$, *where P is "m is even," Q is "m is odd,"* R_1 *is "*$n = 4j + 1$ *for some integer j," and* R_2 *is "*$n = 4i - 1$ *for some integer i." The method we choose is to show that* $P \Rightarrow R_1$ *and* $Q \Rightarrow R_2$.⟩

Case 1. If m is even, then $m = 2j$ for some integer j, and so $n = 2(2j) + 1 = 4j + 1$.
Case 2. If m is odd, then $m = 2k + 1$ for some integer k. In this case, $n = 2(2k + 1) + 1 = 4k + 3 = 4(k + 1) - 1$. Choosing i to be the integer $k + 1$, we have $n = 4i - 1$. ■

The form of proof known as **proof by exhaustion** consists of an examination of every possible case. The statement to be proved may have any form P. For example, to prove that every number x in the closed interval [0, 5] has a certain property, we might consider the cases $x = 0$, $0 < x < 5$, and $x = 5$. The exhaustive method was our method in the example above, and in the proof of Theorem 1.1.1, where we examined all four combinations of truth values for two propositions. Naturally, the idea of proof by exhaustion is appealing only when the number of cases is small, or when large numbers of cases can be systematically handled. Care must be taken to ensure that all possible cases have been considered.

Example. Let x be a real number. Prove that $-|x| \le x \le |x|$.

Proof. ⟨*Since the absolute value of* x *is defined by cases* ($|x| = x$ *if* $x \ge 0$; $|x| = -x$ *if* $x < 0$) *this proof will proceed by cases.*⟩

Case 1. Suppose $x \ge 0$. Then $|x| = x$. Since $x \ge 0$, we have $-x \le x$. Hence, $-x \le x \le x$, which is $-|x| \le x \le |x|$ in this case.

Case 2. Suppose $x < 0$. Then $|x| = -x$. Since $x < 0$, $x \le -x$. Hence, we have $x \le x \le -x$, or $-(-x) \le x \le -x$, which is $-|x| \le x \le |x|$.

Thus, in all cases we have $-|x| \le x \le |x|$. ■

There have been instances of truly exhausting proofs involving great numbers of cases. In 1976, Kenneth Appel and Wolfgang Haken of the University of Illinois announced a proof of the Four-Color Theorem. The original version of their proof of the famous Four-Color Conjecture contains 1,879 cases and took $3\frac{1}{2}$ years to develop.*

Finally, there are proofs by exhaustion with cases so similar in reasoning that we may simply present a single case and alert the reader with the phrase "without loss of generality" that this case represents the essence of arguments for the other cases. Here is an example.

Example. Prove that for the integers m and n, one of which is even and the other odd, $m^2 + n^2$ has the form $4k + 1$ for some integer k.

Proof. Let m and n be integers. Without loss of generality, we may assume that m is even and n is odd. ⟨*The case where m is odd and n is even is similar.*⟩ Then there exist integers s and t such that $m = 2s$ and $n = 2t + 1$. Therefore, $m^2 + n^2 = (2s)^2 + (2t + 1)^2 = 4s^2 + 4t^2 + 4t + 1 = 4(s^2 + t^2 + t) + 1$. Since $s^2 + t^2 + t$ is an integer, $m^2 + n^2$ has the form $4k + 1$ for some integer k. ■

* The Four-Color Theorem involves coloring regions or countries on a map in such a way that no two adjacent countries have the same color. It states that four colors are sufficient, no matter how intertwined the countries may be. The fact that the proof depended so heavily on the computer for checking cases raised questions about the nature of proof. Verifying the 1,879 cases required more than 10 billion calculations. Many people wondered whether there might have been at least one error in a process so lengthy that it could not be carried out by one human being in a lifetime. Haken and Appel's proof has since been improved, and the Four-Color Theorem is accepted; but the debate about the role of computers in proof continues.

Exercises 1.4

1. Analyze the logical form of each of the following statements and construct just the outline of a proof. Since the statements may contain terms with which you are not familiar, you should not (and perhaps could not) provide any details of the proof.
   ★ **(a)** Outline a direct proof that if $(G, *)$ is a cyclic group, then $(G, *)$ is abelian.
   (b) Outline a direct proof that if **B** is a nonsingular matrix, then the determinant of **B** is not zero.
   (c) Suppose A, B, and C are sets. Outline a direct proof that if A is a subset of B and B is a subset of C, then A is a subset of C.
   (d) Outline a direct proof that if the maximum value of the differentiable function f on the closed interval $[a, b]$ occurs at x_0, then either $x_0 = a$ or $x_0 = b$ or $f'(x_0) = 0$.
   (e) Outline a direct proof that if **A** is a diagonal matrix, then **A** is invertible whenever all its diagonal entries are nonzero.
2. A theorem of linear algebra states that if **A** and **B** are invertible matrices, then the product **AB** is invertible. As in Exercise 1, outline
   (a) a direct proof of the theorem.
   (b) a direct proof of the converse of the theorem.
3. Verify that $[(\sim B \Rightarrow M) \wedge \sim L \wedge (\sim M \vee L)] \Rightarrow B$ is a tautology. See the example on page 30.
4. These facts have been established at a crime scene.
   (i) If Professor Plum is not guilty, then the crime took place in the kitchen.
   (ii) If the crime took place at midnight, Professor Plum is guilty.
   (iii) Miss Scarlet is innocent if and only if the weapon was not the candlestick.
   (iv) Either the weapon was the candlestick or the crime took place in the library.
   (v) Either Miss Scarlet or Professor Plum is guilty.

   Use the above and the additional fact(s) below to solve the case. Explain your answer.
   ★ **(a)** The crime lab determines that the crime took place in the library.
   (b) The crime lab determines that the crime did not take place in the library.
   (c) The crime lab determines that the crime was committed at noon with the revolver.
   (d) The crime took place at midnight in the conservatory. (Give a complete answer.)
5. Let x and y be integers. Prove that
   (a) if x and y are even, then $x + y$ is even.
   (b) if x is even, then xy is even.
   (c) if x and y are even, then xy is divisible by 4.
   (d) if x and y are even, then $3x - 5y$ is even.
   (e) if x and y are odd, then $x + y$ is even.

(f) if x and y are odd, then $3x - 5y$ is even.
(g) if x and y are odd, then xy is odd.
★ **(h)** if x is even and y is odd, then $x + y$ is odd.
(i) if x is even and y is odd, then xy is even.

6. Let a and b be real numbers. Prove that
(a) $|ab| = |a||b|$.
(b) $|a - b| = |b - a|$.
(c) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, for $b \neq 0$.
☆ **(d)** $|a + b| \leq |a| + |b|$.
(e) if $|a| \leq b$, then $-b \leq a \leq b$.
(f) if $-b \leq a \leq b$, then $|a| \leq b$.

7. Suppose a, b, c, and d are integers. Prove that
(a) $2a - 1$ is odd.
★ **(b)** if a is even, then $a + 1$ is odd.
(c) if a is odd, then $a + 2$ is odd.
☆ **(d)** $a(a + 1)$ is even.
(e) 1 divides a.
(f) a divides a.
★ **(g)** if a and b are positive and a divides b, then $a \leq b$.
(h) if a divides b, then a divides bc.
★ **(i)** if a and b are positive and $ab = 1$, then $a = b = 1$.
(j) if a and b are positive, a divides b and b divides a, then $a = b$.
(k) if a divides b and c divides d, then ac divides bd.
(l) if ab divides c, then a divides c.
(m) if ac divides bc, then a divides b.

8. Give two proofs that if n is a natural number, then $n^2 + n + 3$ is odd.
(a) Use two cases.
(b) Use Exercises 7(d) and 5(h).

9. Let a, b, and c be integers and x, y, and z be real numbers. Use the technique of working backward from the desired conclusion to prove that
(a) if x and y are nonnegative, then $\frac{x + y}{2} \geq \sqrt{xy}$.
Where in the proof do we use the fact that x and y are nonnegative?
(b) if a divides b and a divides $b + c$, then a divides $3c$.
(c) if $ab > 0$ and $bc < 0$, then $ax^2 + bx + c = 0$ has two real solutions.
(d) if $x^3 + 2x^2 < 0$, then $2x + 5 < 11$.
(e) if an isosceles triangle has sides of length x, y, and z, where $x = y$ and $z = \sqrt{2xy}$, then it is a right triangle.

10. Recall that except for degenerate cases, the graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is

an ellipse iff $B^2 - 4AC < 0$,
a parabola iff $B^2 - 4AC = 0$,
a hyperbola iff $B^2 - 4AC > 0$.

★ **(a)** Prove that the graph of the equation is an ellipse whenever $A > C > B > 0$.

(b) Prove that the graph of the equation is a hyperbola if $AC < 0$ or $B < C < 4A < 0$.

(c) Prove that if the graph is a parabola, then $BC = 0$ or $A = B^2/(4C)$.

Proofs to Grade

11. Exercises throughout the text with this title ask you to examine "Proofs to Grade." These are allegedly true claims and supposed "proofs" of the claims. You should decide the merit of the claim and the validity of the proof and then assign a grade of

A (correct), if the claim and proof are correct, even if the proof is not the simplest or the proof you would have given.

C (partially correct), if the claim is correct *and* the proof is largely correct. The proof may contain one or two incorrect statements or justifications, but the errors are easily correctable.

F (failure), if the claim is incorrect, or the main idea of the proof is incorrect, or there are too many errors.

You must justify assignments of grades other than A and if the proof is incorrect, explain what is incorrect and why.

★ **(a)** Suppose a is an integer.

Claim. If a is odd then $a^2 + 1$ is even.

"Proof." Let a. Then, by squaring an odd we get an odd. An odd plus odd is even. So $a^2 + 1$ is even. ■

(b) Suppose a, b, and c are integers.

Claim. If a divides b and a divides c, then a divides $b + c$.

"Proof." Suppose a divides b and a divides c. Then for some integer q, $b = aq$, and for some integer q, $c = aq$. Then $b + c = aq + aq = 2aq = a(2q)$, so a divides $b + c$. ■

★ **(c)** Suppose x is a positive real number.

Claim. The sum of x and its reciprocal is greater than or equal to 2. That is,

$$x + \frac{1}{x} \geq 2.$$

"Proof." Multiplying by x, we get $x^2 + 1 \geq 2x$. By algebra, $x^2 - 2x + 1 \geq 0$. Thus, $(x - 1)^2 \geq 0$. Any real number squared is greater than or equal to zero, so $x + \frac{1}{x} \geq 2$ is true. ■

★ **(d)** Suppose m is an integer.

Claim. If m^2 is odd, then m is odd.

"Proof." Assume m is odd. Then $m = 2k + 1$ for some integer k. Therefore, $m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. Therefore, if m^2 is odd, then m is odd. ■

(e) Suppose a is an integer.

Claim. $a^3 + a^2$ is even.

"Proof." $a^3 + a^2 = a^2(a + 1)$, which is always an odd number times an even number. Therefore, $a^3 + a^2$ is even. ■