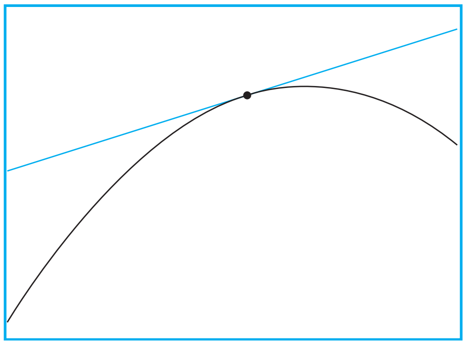
CHAPTER 1



Introduction to Differential Equations

With the systematic study of differential equations, the calculus of functions of a single variable reaches a state of completion. Modeling by differential equations greatly expands the list of possible applications. The list continues to grow as we discover more differential equation models in old and in new areas of application. The use of differential equations makes available to us the full power of the calculus.

When explicit solutions to differential equations are available, they can be used to predict a variety of phenomena. Whether explicit solutions are available or not, we can usually compute useful and very accurate approximate numerical solutions. The use of modern computer technology makes possible the visualization of the results. Furthermore, we continue to discover ways to analyze solutions without knowing the solutions explicitly.

The subject of differential equations is solving problems and making predictions. In this book, we will exhibit many examples of this—in physics, chemistry, and biology, and also in such areas as personal finance and forensics. This is the process of mathematical modeling. If it were not true that differential equations were so useful, we would not be studying them, so we will spend a lot of time on the modeling process and with specific models. In the first section of this chapter we will present some examples of the use of differential equations.

The study of differential equations, and their application, uses the derivative and the integral, the concepts that make up the calculus. We will review these ideas starting in Sections 1.2 and 1.3.

1.1 Differential Equation Models

To start our study of differential equations, we will give a number of examples. This list is meant to be indicative of the many applications of the topic. It is far from being exhaustive. In each case, our discussion will be brief. Most of the examples will be discussed later in the book in greater detail. This section should be considered as advertising for what will be done in the rest of the book.

The theme that you will see in the examples is that in every case we compute the rate of change of a variable in two different ways. First there is the mathematical way. In mathematics, the rate at which a quantity changes is the derivative of that quantity. This is the same for each example. The second way of computing the rate of change comes from the application itself and is different from one application to another. When these two ways of expressing the rate of change are equated, we get a differential equation, the subject we will be studying.

Mechanics

Isaac Newton was responsible for a large number of discoveries in physics and mathematics, but perhaps the three most important are the following:

- The systematic development of the calculus. Newton's achievement was the realization and utilization of the fact that integration and differentiation are operations inverse to each other.
- The discovery of the laws of mechanics. Principal among these was Newton's second law, which says that force acting on a mass (see Figure 1) is equal to the rate of change of momentum with respect to time. Momentum is defined to be the product of mass and velocity, or mv. Thus the force is equal to the derivative of the momentum. If the mass is constant,

$$\frac{d}{dt}mv = m\frac{dv}{dt} = ma,$$

where a is the acceleration. Newton's second law says that the rate of change of momentum is equal to the force F. Expressing the equality of these two ways of looking at the rate of change, we get the equation

$$F = ma, \tag{1.1}$$

the standard expression for Newton's second law.

Figure 1. The force F results in the acceleration of the mass m according to equation (1.1).

- The discovery of the universal law of gravitation. This law says that any body with mass M attracts any other body with mass m directly toward the mass M, with a magnitude proportional to the product of the two masses and inversely proportional to the square of the distance separating them. This means that there is a constant G, which is universal, such that the magnitude of the force is

$$\frac{GMm}{r^2}, \tag{1.2}$$

where r is the distance between the centers of mass of the two bodies. See Figure 2.

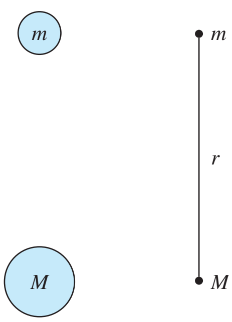


Figure 2. The magnitude of the force attracting two masses is given by equation (1.2).

All of these discoveries were made in the period between 1665 and 1671. The discoveries were presented originally in Newton's *Philosophiae Naturalis Principia Mathematica*, better known as *Principia Mathematica*, published in 1687.

Newton's development of the calculus is what makes the theory and use of differential equations possible. His laws of mechanics create a template for a model for motion in almost complete generality. It is necessary in each case to figure out what forces are acting on a body. His law of gravitation does just that in one very important case.

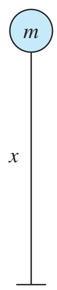


Figure 3. x is the height of the ball above the surface of the earth.

The simplest example is the motion of a ball thrown into the air near the surface of the earth. See Figure 3. If x measures the distance the ball is above the earth, then the velocity and acceleration of the ball are

$$v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

Since the ball is assumed to move only a short distance in comparison to the radius of the earth, the force given by (1.2) may be assumed to be constant. Notice that m, the mass of the ball, occurs in (1.2). We can write the force as $F = -mg$, where $g = GM/r^2$ and r is the radius of the earth. The constant g is called the earth's acceleration due to gravity. The minus sign reflects the fact that the displacement x is measured positively above the surface of the earth, and the force of gravity tends to decrease x. Newton's second law, (1.1), becomes

$$-mg = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}.$$

The masses cancel, and we get the differential equation

$$\frac{d^2x}{dt^2} = -g, \tag{1.3}$$

which is our mathematical model for the motion of the ball.

The equation in (1.3) is called a differential equation because it involves an unknown function $x(t)$ and at least one of its derivatives. In this case the highest derivative occurring is the second order, so this is called a differential equation of second order.

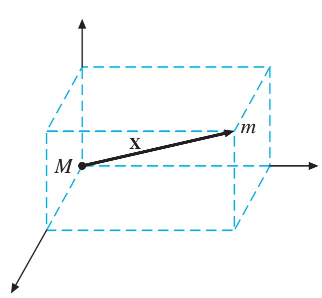


Figure 4. The vector $\mathbf{x}$ gives the location of the planet m relative to the sun M.

A more interesting example of the application of Newton's ideas has to do with planetary motion. For this case, we will assume that the sun with mass M is fixed and put the origin of our coordinate system at the center of the sun. We will denote by $\mathbf{x}(t)$ the vector that gives the location of a planet relative to the sun. See Figure 4. The vector $\mathbf{x}(t)$ has three components. Its derivative is

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt},$$

which is the vector-valued velocity of the planet. For this example, Newton's second law and his law of gravitation become

$$m\frac{d^2\mathbf{x}}{dt^2} = -\frac{GMm}{|\mathbf{x}|^2}\frac{\mathbf{x}}{|\mathbf{x}|}.$$

This system of three second-order differential equations is Newton's model of planetary motion. Newton solved these and verified that the three laws observed by Kepler follow from his model.

Population models

Consider a population $P(t)$ that is varying with time.[1] A mathematician will say that the rate at which the population is changing with respect to time is given by the derivative

$$\frac{dP}{dt}.$$

On the other hand, a population biologist will say that the rate of change is roughly proportional to the population. This means that there is a constant r, called the reproductive rate, such that the rate of change is equal to rP. Putting together the ideas of the mathematician and the biologist, we get the equation

$$\frac{dP}{dt} = rP. \tag{1.4}$$

This is an equation for the function $P(t)$. It involves both P and its derivative, so it is a differential equation. It is not difficult to show by direct substitution into (1.4) that the exponential function

$$P(t) = P_0 e^{rt},$$

where P_0 is a constant representing the initial population, is a solution. Thus, assuming that the reproductive rate r is positive, our population will grow exponentially.

If at this point you go back to the biologist he or she will undoubtedly say that the reproductive rate is not really a constant. While that assumption works for small populations, over the long term you have to take into account the fact that resources of food and space are limited. When you do, a better model for the the reproductive rate is the function $r(1 - P/K)$, and then the rate at which the population changes is better modeled by $r(1 - P/K)P$. Here both r and K are constants.

When we equate our two ideas about the rate at which the population changes, we get the equation

$$\frac{dP}{dt} = r(1 - P/K)P. \tag{1.5}$$

This differential equation for the function $P(t)$ is called the ***logistic equation.*** It is much harder to solve than (1.4), but it does a creditable job of predicting how single populations grow in isolated circumstances.

Pollution

Consider a lake that has a volume of $V = 100$ km^3. It is fed by an input river, and there is another river which is fed by the lake at a rate that keeps the volume of the lake constant. The flow of the input river varies with the season, and assuming that $t = 0$ corresponds to January 1 of the first year of the study, the input rate is

$$r(t) = 50 + 20\cos(2\pi(t - 1/4)).$$

Notice that we are measuring time in years. Thus the maximum flow into the lake occurs when $t = 1/4$, or at the beginning of April.

In addition, there is a factory on the lake that introduces a pollutant into the lake at the rate of 2 km^3/year. Let $x(t)$ denote the total amount of pollution in the lake

[1] For the time being, the population can be anything—humans, paramecia, butterflies, and so on. We will be more careful later.

at time t. If we make the assumption that the pollutant is rapidly mixed throughout the lake, then we can show that $x(t)$ satisfies the differential equation

$$\frac{dx}{dt} = 2 - \big(52 + 20\cos(2\pi(t - 1/4))\big)\frac{x}{100}.$$

This equation can be solved and we can then answer questions about how dangerous the pollution problem really is. For example, if we know that a concentration of less than 2% is safe, will there there be a problem? The solution will tell us.

The assumption that the pollutant is rapidly mixed into the lake is not very realistic. We know that this does not happen, especially in this situation, where there is a flow of water through the lake. This assumption can be removed, but to do so, we need to allow the concentration of the pollutant to vary with position in the lake as well as with time. Thus the concentration is a function $c(t, x, y, z)$, where (x, y, z) represents a position in the three-dimensional lake. Instead of assuming perfect mixing, we will assume that the pollutant diffuses through water at a certain rate. Once again we can construct a mathematical model. Again it will be a differential equation, but now it will involve partial derivatives with respect to the spatial coordinates x, y, and z, as well as the time t.

Personal finance

How much does a person need to save during his or her work life in order to be sure of a retirement without money worries? How much is it necessary to save each year in order to accumulate these assets? Suppose one's salary increases over time. What percent of one's salary should be saved to reach one's retirement goal?

All of these questions, and many more like them, can be modeled using differential equations. Then, assuming particular values for important parameters like return on investment and rate of increase of one's salary, answers can be found.

Other examples

We have given four examples. We could have given a hundred more. We could talk about electrical circuits, the behavior of musical instruments, the shortest paths on a complicated-looking surface, finding a family of curves that are orthogonal to a given family, discovering how two coexisting species interact, and many others.

All of these examples use ordinary differential equations. The applications of partial differential equations go much farther. We can include electricity and magnetism; quantum chromodynamics, which unifies electricity and magnetism with the weak and strong nuclear forces, the flow of heat, oscillations of many kinds, such as vibrating strings, the fair pricing of stock options, and many more.

The use of differential equations provides a way to reduce many areas of application to mathematical analysis. In this book, we will learn how to do the modeling and how to use the models after we make them.

EXERCISES

The phrase "y is proportional to x" implies that y is related to x via the equation $y = kx$, where k is a constant. In a similar manner, "y is proportional to the square of x" implies $y = kx^2$, "y is proportional to the product of x and z" implies $y = kxz$, and "y is inversely proportional to the cube of x" implies $y = k/x^3$. For example, when Newton proposed that the force of attraction of one body on another is proportional to the product of the masses and inversely proportional to the square of the distance between them, we can immediately write

$$F = \frac{GMm}{r^2},$$

where G is the constant of proportionality, usually known as the universal gravitational constant. In Exercises 1–11, use these ideas to model each application with a differential equation. All rates are assumed to be with respect to time.

1. The rate of growth of bacteria in a petri dish is proportional to the number of bacteria in the dish.
2. The rate of growth of a population of field mice is inversely proportional to the square root of the population.
3. A certain area can sustain a maximum population of 100 ferrets. The rate of growth of a population of ferrets in this area is proportional to the product of the population and the difference between the actual population and the maximum sustainable population.
4. The rate of decay of a given radioactive substance is proportional to the amount of substance remaining.
5. The rate of decay of a certain substance is inversely proportional to the amount of substance remaining.
6. A potato that has been cooking for some time is removed from a heated oven. The room temperature of the kitchen is 65°F. The rate at which the potato cools is proportional to the difference between the room temperature and the temperature of the potato.
7. A thermometer is placed in a glass of ice water and allowed to cool for an extended period of time. The thermometer is removed from the ice water and placed in a room having temperature 77°F. The rate at which the thermometer warms is proportional to the difference in the room temperature and the temperature of the thermometer.
8. A particle moves along the x-axis, its position from the origin at time t given by $x(t)$. A single force acts on the particle that is proportional to, but opposite the object's displacement. Use Newton's law to derive a differential equation for the object's motion.
9. Use Newton's law to develop the equation of motion for the particle in Exercise 8 if the force is proportional to, but opposite the square of the particle's velocity.
10. Use Newton's law to develop the equation of motion for the particle in Exercise 8 if the force is inversely proportional to, but opposite the square of the particle's displacement from the origin.
11. The voltage drop across an inductor is proportional to the rate at which the current is changing with respect to time.

1.2 The Derivative

Before reading this section, ask yourself, "What is the derivative?" Several answers may come to mind, but remember your first answer.

Chances are very good that your answer was one of the following five:

1. The rate of change of a function
2. The slope of the tangent line to the graph of a function
3. The best linear approximation of a function
4. The limit of difference quotients,

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

5. A table containing items such as we see in Table 1

Table 1 A table of derivatives

$f(x) =$	$f'(x) =$
C	0
x	1
x^n	nx^{n-1}
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$
e^x	e^x
$\ln(\|x\|)$	$1/x$

All of these answers are correct. Each of them provides a different way of looking at the derivative. The best answer to the question is "all of the above." Since we will be using all five ways of looking at the derivative, let's spend a little time discussing each.

The rate of change

In calculus, we learn that a function has an instantaneous rate of change, and this rate is equal to the derivative. For example, if we have a distance $x(t)$ measured from a fixed point on a line, then the rate at which x changes with respect to time is the velocity v. We know that

$$v = x' = \frac{dx}{dt}.$$

Similarly, the acceleration a is the rate of change of the velocity, so

$$a = v' = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

These facts about linear motion are reflected in many other fields. For example, in economics, the law of supply and demand says that the price of a product is determined by the supply of that product and the demand for it. If we assume that the demand is constant, then the price P is a function of the supply S, or $P = P(S)$. The rate at which P changes with the supply is called the marginal price. In mathematical terms, the marginal price is simply the derivative $P' = dP/dS$. We can also talk about the rate of change of the mass of a radioactive material, of the size of population, of the charge on a capacitor, of the amount of money in a savings account or an investment account, or of many more quantities.[2]

We will see all of these examples and more in this book. The point is that when any quantity changes, the rate at which it changes is the derivative of that quantity. It is this fact that starts the modeling process and makes the study of differential equations so useful. For this reason we will refer to the statement that the derivative is the rate of change as the ***modeling definition*** of the derivative.

The slope of the tangent line

This provides a good way to visualize the derivative. Look at Figure 1. There you see the graph of a function f, and the tangent line to the graph of f at the point $(x_0, f(x_0))$. The equation of the tangent line is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

From this formula, it is easily seen that the slope of the tangent line is $f'(x_0)$.

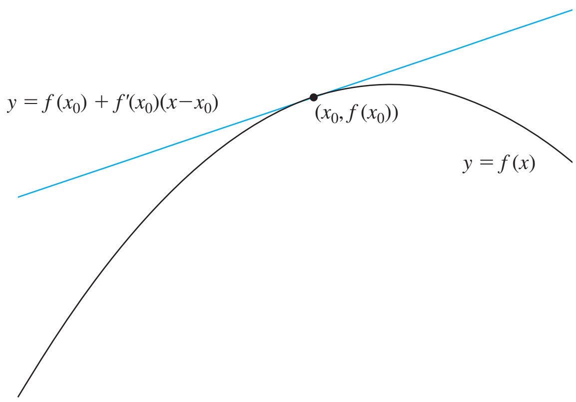


Figure 1. The derivative is the slope of the tangent line to the graph of the function.

Again looking at Figure 1, we can visualize the rate at which the function f is changing as x changes near the point x_0. It is the same as the slope of the tangent line.

We will refer to this characterization of the derivative as the ***geometric definition*** of the derivative.

[2] In all but one of the mentioned examples, the quantity changes with respect to time. Most of the applications of ordinary differential equations involve rates of change with respect to time. For this reason, t is usually used as the independent variable. However, there are cases where things change depending on other parameters, as we will see. Where appropriate, we will use other letters to denote the independent variable. Sometimes we will do so just for practice.

The best linear approximation

Let

$$L(x) = f(x_0) + f'(x_0)(x - x_0). \tag{2.1}$$

L is a linear (or affine) function of x. Taylor's theorem says there is a remainder function $R(x)$, such that

$$f(x) = L(x) + R(x) \quad \text{and} \quad \lim_{x \to x_0} \frac{R(x)}{x - x_0} = 0. \tag{2.2}$$

The limit in (2.2) means that $R(x)$ gets small as $x \to x_0$. In fact, it gets enough smaller than $x - x_0$ that the ratio goes to 0. It turns out that the function L defined in (2.1) is the only linear function with this property. This is what we mean when we say that L is the best linear approximation to the nonlinear function f. You will also notice that the straight line in Figure 1 is the graph of L. In fact, Figure 1 provides a pictorial demonstration that $L(x)$ is a good approximation for $f(x)$ for x near x_0.

The formula in (2.1) defines $L(x)$ in terms of the derivative of f. In this sense, the derivative gives us the best linear approximation to the nonlinear function f near $x = x_0$. [Actually (2.1) contains three important pieces of data, x_0, $f(x_0)$, and $f'(x_0)$. We are perhaps stretching the point when we say that it is the derivative alone that enables us to find a linear approximation to f, but it is clear that the derivative is the most important of these three.]

Since the linear approximation is an algebraic object, we will refer to this as the ***algebraic definition*** of derivative.

The limit of difference quotients

Consider the difference quotient

$$m = \frac{f(x) - f(x_0)}{x - x_0}. \tag{2.3}$$

This is equal to the slope of the line through the two points $(x_0, f(x_0))$ and $(x, f(x))$ as illustrated in Figure 2. We will refer to this line as a secant line. As x approaches x_0, the secant line approaches the tangent line shown in Figure 1. This is reflected in the fact that

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{2.4}$$

Thus the slope of the tangent line, $f'(x_0)$, is the limit of the slopes of secant lines.

The difference quotient in (2.3) is also the average rate of change of the function f between x_0 and x. As the interval between x_0 and x is made smaller, these average rates approach the instantaneous rate of change of f. Thus we see the connection with our modeling definition.

The definition of the derivative given in (2.4) will be called the ***limit quotient definition***. This is the definition that most mathematicians think of when asked to define the derivative. However, as we will see, it is also very useful, even when attempting to find mathematical models.

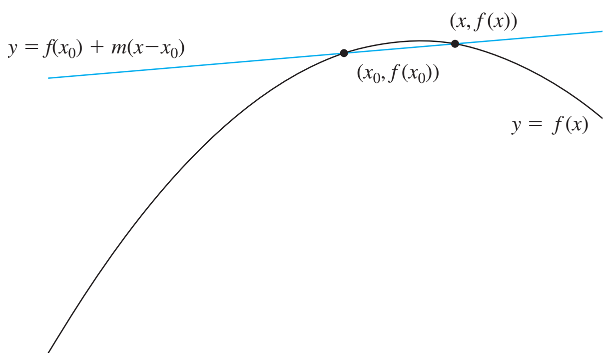


Figure 2. The secant line with slope m given by the difference quotient in (2.3).

The table of formulas

By memorizing a table of derivatives and a few formulas (especially the chain rule), we can learn the skill of differentiation. It isn't hard to be confident that you can compute the derivative of any given function. This skill is important. However, it is clear that this ***formulaic definition*** of derivative is quite different from those given previously.

A complete understanding of the formulaic definition is important, but it does not provide any information about the other definitions we have examined. Therefore, it helps us neither to apply the derivative in modeling nature nor to understand its properties. For that reason, the formulaic definition is incomplete. This is not true of the other definitions. Starting with one of them, it is possible to construct a table that will give us the formulaic finesse we need. Admittedly that is a big task. That was what was done (or should have been done) in your first calculus course.

To sum up, we have examined five definitions of the derivative. Each of these emphasizes a different aspect or property of the derivative. All of them are important. We will see this as we progress through the study of differential equations. If your answer to the question at the beginning of the section was any of these five, your answer is correct. However, a complete understanding of the derivative requires the understanding of all five definitions.

Even if your answer was not on the list of five, it may be correct. The famous mathematician William Thurston once compiled a list of over 40 "definitions" of the derivative. Of course many of these appear only in more advanced parts of mathematics, but the point is made that the derivative appears in many ways in mathematics and in its applications. It is one of the most fundamental ideas in mathematics and in its application to science and technology.

EXERCISES

You might recall the following rules of differentiation from your calculus class. Let f and g be differentiable functions of x. Then

$$(cf)' = cf' \qquad (f \pm g)' = f' \pm g'$$
$$(fg)' = f'g + fg' \qquad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Also, the chain rule is essential.

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

Use these rules, plus the table of derivatives in Table 1, to find the derivative of each of the functions in Exercises 1–12.

1. $f(x) = 3x - 5$

2. $f(x) = 5x^2 - 4x - 8$

3. $f(x) = 3\sin 5x$

4. $f(x) = \cos 2\pi x$

5. $f(x) = e^{3x}$

6. $f(x) = 5e^{x^2}$

7. $f(x) = \ln|5x|$

8. $f(x) = \ln(\cos 2x)$

9. $f(x) = x\ln x$

10. $f(x) = e^x \sin\pi x$

11. $f(x) = \dfrac{x^2}{\ln x}$ **12.** $f(x) = \dfrac{x \ln x}{\cos x}$

13. Suppose that f is differentiable at x_0. Let L be the "best linear approximation" defined by $L(x) = f(x_0) + f'(x_0)(x - x_0)$. Given that $R(x) = f(x) - L(x)$, show that

$$\lim_{x \to x_0} \frac{R(x)}{x - x_0} = 0.$$

For each of the functions given in Exercises 14–17, sketch the function f and its linearization $L(x) = f(x_0) + f'(x_0)(x - x_0)$ at the given point x_0 on the same set of coordinate axes.

14. $f(x) = e^x$, at $x_0 = 0$

15. $f(x) = \cos x$, at $x_0 = \pi/4$

16. $f(x) = \sqrt{x}$, at $x_0 = 1$

17. $f(x) = \ln(1 + x)$, at $x_0 = 0$

In order that $R(x)/(x - x_0)$ of equation (2.2) approach zero as $x \to x_0$, the numerator $R(x)$ must approach zero at a faster rate than does the denominator $x - x_0$. For each of Exercises 18–21, sketch the graph of $y = x - x_0$ and $R(x) = f(x) - L(x)$ on the same set of coordinate axes. Do both $x - x_0$ and $R(x)$ approach zero as $x \to x_0$? Which approaches zero at a faster rate, $R(x)$ or $x - x_0$?

18. $f(x) = x^{3/2}$, at $x_0 = 1$

19. $f(x) = \sin 2x$, at $x = \pi/8$

20. $f(x) = \sqrt{x + 1}$, at $x = 0$

21. $f(x) = x e^{x-1}$, at $x = 1$

1.3 Integration

We can start once more by asking the question, "What is the integral?" This time our list of possible answers is not so long.

1. The area under the graph of a function
2. The antiderivative
3. A table containing items such as we see in Table 1

Table 1 A table of integrals

$f(x) =$	$\int f(x)\,dx =$	$f(x) =$	$\int f(x)\,dx =$
0	C	$\cos(x)$	$\sin(x) + C$
1	$x + C$	$\sin(x)$	$-\cos(x) + C$
x	$\dfrac{x^2}{2} + C$	e^x	$e^x + C$
x^n	$\dfrac{x^{n+1}}{n+1} + C$	$\dfrac{1}{x}$	$\ln(\lvert x \rvert) + C$

Let's look at each of them briefly.

The area under the graph

The first answer emphasizes the definite integral. The definite integral

$$\int_a^b f(x)\,dx \tag{3.1}$$

is interpreted as the area under the graph of the function f between $x = a$ and $x = b$. It represents the area of the shaded region in Figure 1.

This is the most fundamental definition of the integral. The integral was invented to solve the problem of finding the area of regions that are not simple rectangles or circles. Despite its origin as a method to use in this one application, it has found numerous other applications.

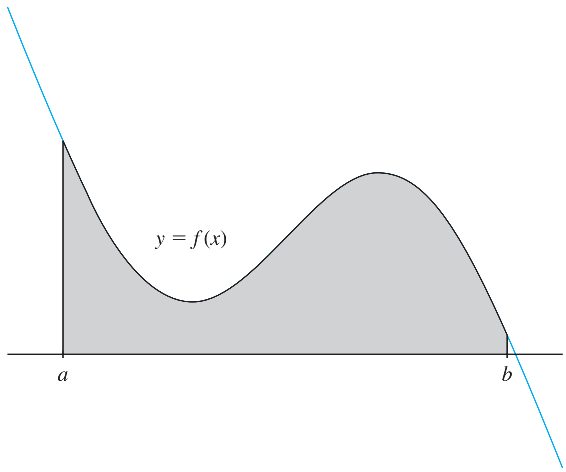


Figure 1. The area of the shaded region is the integral in (3.1).

The antiderivative

This answer emphasizes the indefinite integral. In fact, the phrase ***indefinite integral*** is a synonym for ***antiderivative.*** The definition is summed up in the following equivalence. If the function g is continuous, then

$$f' = g \quad \text{if and only if} \quad \int g(x)\,dx = f(x) + C. \tag{3.2}$$

In (3.2), C refers to the arbitrary constant of integration. Thus the process of indefinite integration involves finding antiderivatives. Given a function g, we want to find a function f such that $f' = g$.

The connection between the definite and the indefinite integral is found in the ***fundamental theorem of calculus***. This says that if $f' = g$, then

$$\int_a^b g(x)\,dx = f(b) - f(a).$$

The table of formulas

This formulaic approach to the integral has the same features and failures as the formulaic approach to the derivative. It leads to the handy skill of integration, but it does not lead to any deep understanding of the integral.

All of these approaches to the integral are important. It is very important to understand the first two and how they are connected by the fundamental theorem. However, for the elementary part of the study of ordinary differential equations, it is really the second and third approaches that are most important. In other words, it is important to be able to find antiderivatives.

Solution by integration

The solution of an important class of differential equations amounts to finding antiderivatives. A first-order differential equation can be written as

$$y' = f(t, y), \tag{3.3}$$

where the right-hand side is a function of the independent variable t and the unknown function y. Suppose that the right-hand side is a function only of t and does not depend on y. Then equation (3.3) becomes

$$y' = f(t).$$

Comparing this with (3.2), we see immediately that the solution is

$$y(t) = \int f(t)\,dt. \tag{3.4}$$

Let's look at an example.

Example 3.5 Solve the differential equation

$$y' = \cos t. \tag{3.6}$$

According to (3.4), the solution is

$$y(t) = \int \cos(t)\,dt = \sin t + C, \tag{3.7}$$

where C is an arbitrary constant. That's pretty easy. It is just the process of integration. It's old hat to you by now. Solving the more general equation in (3.3) is not so easy, as we will see.

The constant of integration C makes (3.7) a one-parameter family of solutions of (3.6) defined on $(-\infty, \infty)$. This is an example of a ***general solution*** to a differential equation. Some of these solutions are drawn in Figure 2. ●

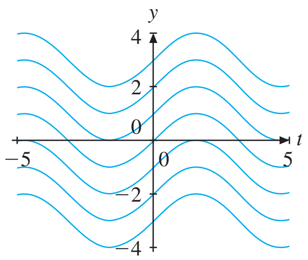


Figure 2. Several solutions to (3.6).

It is significant that the solution curves of equation (3.6) shown in Figure 2 are vertical translates of one another. That is to say, any solution curve can be obtained from any other by a vertical translation. This is always the case for solution curves of an equation of the form $y' = f(t)$. According to (3.2), if $y(t) = F(t)$ is one solution to the equation, then all others are of the form $y(t) = F(t) + C$ for some constant C. The graphs of such functions are vertical translates of the graph of $y(t) = F(t)$.

The constant of integration allows us to put an extra condition on a solution. This is illustrated in the next example.

Example 3.8 Find the solution to $y'(t) = te^t$ that satisfies $y(0) = 2$.

This is an example of an ***initial value problem***. It requires finding the particular solution that satisfies the ***initial condition*** $y(0) = 2$. According to (3.2), the general solution to the differential equation is given by

$$y(t) = \int te^t\,dt. \tag{3.9}$$

This integral can be evaluated using integration by parts. Since this method is so useful, we will briefly review it. In general, it says

$$\int u\,dv = uv - \int v\,du, \tag{3.10}$$

where u and v are functions. If they are functions of t, then $du = u'(t)\,dt$ and $dv = v'(t)\,dt$. For the integral in equation (3.9), we let $u(t) = t$, and $dv = v'(t)\,dt = e^t\,dt$. Then $du = dt$ and $v(t) = e^t$, and equation (3.10) gives

$$\int te^t\,dt = \int u\,dv = uv - \int v\,du = te^t - \int e^t\,dt.$$

After evaluating the last integral, we see that

$$y(t) = te^t - e^t + C = e^t(t-1) + C. \tag{3.11}$$

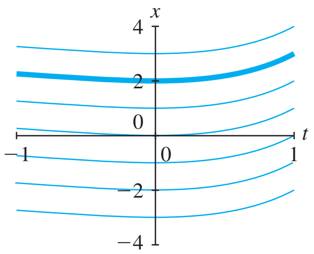


Figure 3. The solution of the initial value problem in Example 3.8 passes through the point (0, 2).

This one-parameter family of solutions is the general solution to the equation $y' = te^t$. Each member of the family exists on the interval $(-\infty, \infty)$. The condition $y(0) = 2$ can be used to determine the constant C.

$$2 = y(0) = e^0(0-1) + C = -1 + C$$

Therefore, $C = 3$ and the solution of the initial value problem is

$$y(t) = e^t(t-1) + 3. \tag{3.12}$$

It is important to note that the solution curve defined by equation (3.12) is the member of the family of solution curves defined by (3.11) that passes through the point (0, 2), as shown in Figure 3. ●

The use of initial conditions to determine a particular solution can be affected from the beginning of the solution process by using definite integrals instead of indefinite integrals. For example, in Example 3.8, we can proceed using the fundamental theorem of calculus:

$$y(t) - y(0) = \int_0^t y'(u)\,du.$$

Hence,

$$\begin{aligned} y(t) &= y(0) + \int_0^t ue^u\,du \\ &= 2 + ue^u - e^u\Big|_0^t \\ &= e^t(t-1) + 3. \end{aligned}$$

We will not always use the letter t to designate the independent variable. Any letter will do, as long as we are consistent. The same is true of the dependent variable.

Example 3.13 Find the solution to the initial value problem

$$y' = \frac{1}{x} \quad \text{with} \quad y(1) = 3.$$

Here we are using x as the independent variable. By integration, we find that

$$y(x) = \ln(|x|) + C.$$

We are asked for the solution that satisfies the initial condition

$$3 = y(1) = \ln(1) + C = C.$$

Thus, $C = 3$.

A solution to a differential equation has to have a derivative at every point. Therefore, it is also continuous. However, the function $y(x) = \ln(|x|) + 3$ is not defined for $x = 0$. To get a continuous function from y, we have to limit its domain to $(0, \infty)$ or $(-\infty, 0)$. Since we want a solution that is defined at $x = 1$, we must choose $(0, \infty)$. Thus, our solution is

$$y(x) = \ln(x) + 3 \quad \text{for} \quad x > 0.$$ ●

The motion of a ball

In Section 1.1, we talked about the application of Newton's laws to the motion of a ball near the surface of the earth. See Figure 4. The model we derived [in equation (1.3)] was

$$\frac{d^2x}{dt^2} = -g,$$

where $x(t)$ is the height of the ball above the surface of the earth and g is the acceleration due to gravity. If we measure x in feet and time in seconds, $g = 32$ ft/s^2.

Example 3.14 Suppose a ball is thrown into the air with initial velocity $v_0 = 20$ ft/s. Assuming the ball is thrown from a height of $x_0 = 6$ feet, how long does it take for the ball to hit the ground?

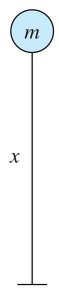


Figure 4. x is the height of the ball above the surface of the earth.

We can solve this equation using the methods of this section. First we introduce the velocity to reduce the second-order equation to a system of two first-order equations:

$$\frac{dx}{dt} = v, \quad \text{and} \quad \frac{dv}{dt} = -g. \tag{3.15}$$

Solving the second equation by integration, we get

$$v(t) = -gt + C_1.$$

Evaluating this at $t = 0$, we see that the constant of integration is $C_1 = v(0) = v_0 = 20$, the initial velocity. Hence, the velocity is $v(t) = -gt + v_0 = -32t + 20$, and the first equation in (3.15) becomes

$$\frac{dx}{dt} = -gt + v_0 = -32t + 20.$$

Solving by integration, we get

$$x(t) = -\frac{1}{2}gt^2 + v_0t + C_2 = -16t^2 + 20t + C_2.$$

Once more we evaluate this at $t = 0$ to show that $C_2 = x(0) = x_0 = 6$, the initial elevation of the ball. Hence, our final solution is

$$x(t) = -\frac{1}{2}gt^2 + v_0t + x_0 = -16t^2 + 20t + 6. \tag{3.16}$$

The ball hits the ground when $x(t) = 0$. By solving the quadratic equation, and using the positive solution, we see that the ball hits the ground after 1.5 seconds. ●

EXERCISES

In Exercises 1–8, find the general solution of the given differential equation. In each case, sketch at least six members of the family of solution curves.

1. $y' = 2t + 3$

2. $y' = 3t^2 + 2t + 3$

3. $y' = \sin 2t + 2\cos 3t$

4. $y' = 2\sin 3t - \cos 5t$

5. $y' = \dfrac{t}{1 + t^2}$

6. $y' = \dfrac{3t}{1 + 2t^2}$

7. $y' = t^2 e^{3t}$

8. $y' = t\cos 3t$

In Exercises 1–8 above, each equation has the form $y' = f(t, y)$, the goal being to find a solution $y = y(t)$; that is, find y as a function of t. Of course, you are free to choose different letters, both for the dependent and independent variables. For example, in the differential equation $s' = xe^x$, it is understood that $s' = ds/dx$, and the goal is to find a solution s as a function of x; that is, $s = s(x)$. In Exercises 9–14, find the general solution of the given differential equation. In each case, sketch at least six members of the family of solution curves.

9. $s' = e^{-2\omega} \sin \omega$

10. $y' = x \sin 3x$

11. $x' = s^2 e^{-s}$

12. $s' = e^{-u} \cos u$

13. $r' = \dfrac{1}{u(1-u)}$

14. $y' = \dfrac{3}{x(4-x)}$

Note: Exercises 13 and 14 require a partial fraction decomposition. If you have forgotten this technique, you can find extensive explanation in Section 5.3 of this text. In particular, see Example 3.6 in that section.

In Exercises 15–24, find the solution of each initial value problem. In each case, sketch the solution.

15. $y' = 4t - 6, \quad y(0) = 1$

16. $y' = x^2 + 4, \quad y(0) = -2$

17. $x' = te^{-t^2}, \quad x(0) = 1$

18. $r' = t/(1 + t^2), \quad r(0) = 1$

19. $s' = r^2 \cos 2r, \quad s(0) = 1$

20. $P' = e^{-t} \cos 4t, \quad P(0) = 1$

21. $x' = \sqrt{4 - t}, \quad x(0) = 1$

22. $u' = 1/(x - 5), \quad u(0) = -1$

23. $y' = \dfrac{t+1}{t(t+4)}, \quad y(-1) = 0$

24. $v' = \dfrac{r^2}{r+1}, \quad v(0) = 0$

In Exercises 25–28, assume that the motion of a ball takes place in the absence of friction. That is, the only force acting on the ball is the force due to gravity.

25. A ball is thrown into the air from an initial height of 3 m with an initial velocity of 50 m/s. What is the position and velocity of the ball after 3 s?

26. A ball is dropped from rest from a height of 200 m. What is the velocity and position of the ball 3 seconds later?

27. A ball is thrown into the air from an initial height of 6 m with an initial velocity of 120 m/s. What will be the maximum height of the ball and at what time will this event occur?

28. A ball is propelled downward from an initial height of 1000 m with an initial speed of 25 m/s. Calculate the time that the ball hits the ground.