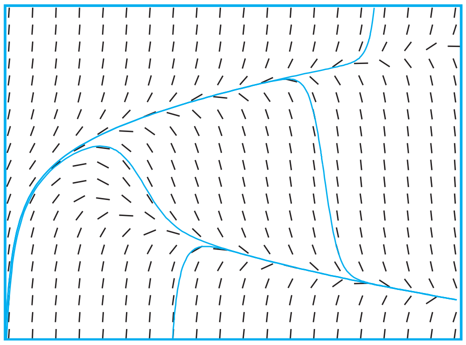
CHAPTER 2



First-Order Equations

In this chapter, we will study first-order equations. We will begin in Section 2.1 by making some definitions and presenting an overview of what we will cover in this chapter. We will then alternate between methods of finding exact solutions and some applications that can be studied using those methods. For each application, we will carefully derive the mathematical models and explore the existence of exact solutions. We will end by showing how qualitative methods can be used to derive useful information about the solutions.

2.1 Differential Equations and Solutions

In this section, we will give an overview of what we want to learn in this chapter. We will visit each topic briefly to give a flavor of what will follow in succeeding sections.

Ordinary differential equations

An ***ordinary differential equation*** is an equation involving an unknown function of a single variable together with one or more of its derivatives. For example, the equation

$$\frac{dy}{dt} = y - t \tag{1.1}$$

is an ordinary differential equation. Here $y = y(t)$ is the unknown function and t is the ***independent variable***.

Some other examples of ordinary differential equations are

$$
\begin{aligned}
y' &= y^2 - t & ty' &= y \\
y' + 4y &= e^{-3t} & y' &= \cos(ty) \\
yy'' + t^2 y &= \cos(t) & y'' &= y^2.
\end{aligned}
\tag{1.2}
$$

The ***order*** of a differential equation is the order of the highest derivative that occurs in the equation. Thus the equation in (1.1) is a ***first-order*** equation since it involves only the first derivative of the unknown function. All of the equations listed in the first two rows of (1.2) are first order. Those in the third row are ***second order*** because they involve the second derivative of y.

The equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \tag{1.3}$$

is not an ordinary differential equation, since the unknown function w is a function of the two independent variables t and x. Because it involves partial derivatives of an unknown function of more than one independent variable, equation (1.3) is called a ***partial differential equation***. For the time being we are interested only in ordinary differential equations.

Normal form

Any first order equation can be put into the form

$$\phi(t, y, y') = 0, \tag{1.4}$$

where ϕ is a function of three variables. For example, the equation in (1.1) can be written as

$$y' - y - t = 0.$$

This equation has the form in (1.4) with $\phi(t, y, z) = z - y - t$. Similarly, the general equation of order n can be written as

$$\phi(t, y, y', \dots, y^{(n)}) = 0, \tag{1.5}$$

where ϕ is a function of $n + 1$ variables. Notice that all of the equations in (1.2) can be put into this form.

The general forms in (1.4) and (1.5) are too general to deal with in many instances. Frequently we will find it useful to solve for the highest derivative. We will give the result a name.

DEFINITION 1.6 A first-order differential equation of the form

$$y' = f(t, y)$$

is said to be in ***normal form***. Similarly, an equation of order n having the form

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

is said to be in ***normal form***.

Example 1.7 Place the differential equation $t + 4yy' = 0$ into normal form.

This is accomplished by solving the equation $t + 4yy' = 0$ for y'. We find that

$$y' = -\frac{t}{4y}. \tag{1.8}$$

Note that the right-hand side of equation (1.8) is a function of t and y, as required by the normal form $y' = f(t, y)$.

Solutions

A ***solution*** of the first-order, ordinary differential equation $\phi(t, y, y') = 0$ is a differentiable function $y(t)$ such that $\phi(t, y(t), y'(t)) = 0$ for all t in the interval[1] where $y(t)$ is defined.

To discover if a given function is a solution to a differential equation we substitute the function and its derivative(s) into the equation. For example, we can show that $y(t) = t + 1$ is a solution to equation (1.1) by substitution. It is only necessary to compute both sides of equation (1.1) and show that they are equal. We have

$$y'(t) = 1, \quad \text{and} \quad y(t) - t = t + 1 - t = 1.$$

Since the left- and right-hand sides are equal, $y(t) = t + 1$ is a solution.

The process of verifying that a given function is or is not a solution to a differential equation is a very important skill. You can use it to check that your homework solutions are correct. We will use it repeatedly for a variety of purposes, including finding solution methods. Here are two more examples.

Example 1.9 Show that $y(t) = Ce^{-t^2}$ is a solution of the first-order equation

$$y' = -2ty, \tag{1.10}$$

where C is an arbitrary real number.

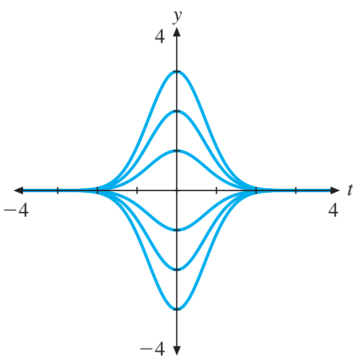


Figure 1. Several solutions to equation (1.10).

We compute both sides of the equation and compare them. On the left, we have $y'(t) = -2tCe^{-t^2}$, and on the right, $-2ty(t) = -2tCe^{-t^2}$, so the equation is satisfied. Both $y(t)$ and $y'(t)$ are defined on the interval $(-\infty, \infty)$. Therefore, for each real number C, $y(t) = Ce^{-t^2}$ is a solution of equation (1.10) on the interval $(-\infty, \infty)$.

Example 1.9 illustrates the fact that a differential equation can have lots of solutions. The solution formula $y(t) = Ce^{-t^2}$ gives a different solution for very value of the constant C. We will see in Section 2.4 that every solution to equation (1.10) is of this form for some value of the constant C. For this reason the formula $y(t) = Ce^{-t^2}$ is called the ***general solution*** to (1.10). The graphs of these solutions are called ***solution curves***, several of which are drawn in Figure 1.

Example 1.11 Is the function $y(t) = \cos t$ a solution to the differential equation $y' = 1 + y^2$?

We substitute $y(t) = \cos t$ into the equation. On the left-hand side we have $y' = -\sin t$. On the right-hand side, $1 + y^2 = 1 + \cos^2 t$. Since $-\sin t \neq 1 + \cos^2 t$ for most values of t, $y(t) = \cos t$ is not a solution.

[1] We will use the notation (a, b), $[a, b]$, $(a, b]$, $[a, b)$, (a, ∞), $[a, \infty)$, $(-\infty, b)$, $(-\infty, b]$, and $(-\infty, \infty)$ for intervals. For example, $(a, b) = \{t : a < t < b\}$, $[a, b) = \{t : a \le t < b\}$, $(-\infty, b] = \{t : t \le b\}$, and so on.

Initial value problems

In Example 1.9, we found a general solution, indicated by the presence of an undetermined constant in the formula. This reflects the fact that an ordinary differential equation has infinitely many solutions. In applications, it is necessary to use other information, in addition to the differential equation, to determine the value of the constant and to specify the solution completely. Such a solution is called a ***particular solution***.

Example 1.12 Given that

$$y(t) = -\frac{1}{t - C} \tag{1.13}$$

is a general solution of $y' = y^2$, find the particular solution satisfying $y(0) = 1$.

Because

$$1 = y(0) = \frac{-1}{0 - C} = \frac{1}{C},$$

$C = 1$. Substituting $C = 1$ in equation (1.13) makes

$$y(t) = -\frac{1}{t - 1}, \tag{1.14}$$

a particular solution of $y' = y^2$, satisfying $y(0) = 1$. ●

DEFINITION 1.15 A first-order differential equation together with an initial condition,

$$y' = f(t, y), \quad y(t_0) = y_0, \tag{1.16}$$

is called an ***initial value problem***. A solution of the initial value problem is a differentiable function $y(t)$ such that

1. $y'(t) = f(t, y(t))$ for all t in an interval containing t_0 where $y(t)$ is defined, and
2. $y(t_0) = y_0$.

Thus, in Example 1.12, the function $y(t) = 1/(1 - t)$ is the solution to the initial value problem

$$y' = y^2, \quad \text{with} \quad y(0) = 1.$$

Interval of existence

The ***interval of existence*** of a solution to a differential equation is defined to be the largest interval over which the solution can be defined and remain a solution. It is important to remember that solutions to differential equations are required to be differentiable, and this implies that they are continuous. The solution to the initial value problem in Example 1.12 is revealing.

Example 1.17 Find the interval of existence for the solution to the initial value problem

$$y' = y^2 \quad \text{with} \quad y(0) = 1.$$

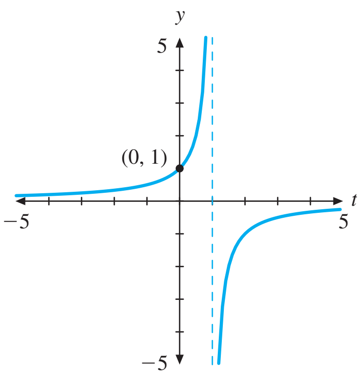


Figure 2. The graph of $y = -1/(t-1)$.

In Example 1.12, we found that the solution is

$$y(t) = \frac{-1}{t-1}.$$

The graph of y is a hyperbola with two branches, as shown in Figure 2. The function y has an infinite discontinuity at $t = 1$. Consequently, this function cannot be considered to be a solution to the differential equation $y' = y^2$ over the whole real line.

Note that the left branch of the hyperbola in Figure 2 passes through the point $(0, 1)$, as required by the initial condition $y(0) = 1$. Hence, the left branch of the hyperbola is the solution curve needed. This particular solution curve extends indefinitely to the left, but rises to positive infinity as it approaches the asymptote $t = 1$ from the left. Any attempt to extend this solution to the right would have to include $t = 1$, at which point the function $y(t)$ is undefined. Consequently, the maximum interval on which this solution curve is defined is the interval $(-\infty, 1)$. This is the interval of existence. ●

Using variables other than y and t

So far all of our examples have used y as the unknown function, and t as the independent variable. It is not required to use y and t. We can use any letter to designate the independent variable and any other for the unknown function. For example, the equation

$$y' = x + y$$

has the form $y' = f(x, y)$, making x the independent variable and requiring a solution y that is a function of x. This equation has general solution

$$y(x) = -1 - x + Ce^x,$$

which exists on $(-\infty, \infty)$.

Similarly, in the equation

$$s' = \sqrt{r},$$

the independent variable is r and the unknown function is s, so s must be a function of r. The general solution of this equation is

$$s(r) = \frac{2}{3}r^{3/2} + C.$$

This general solution exists on the interval $[0, \infty)$.

Example 1.18 Verify that $x(s) = 2 - Ce^{-s}$ is a solution of

$$x' = 2 - x \tag{1.19}$$

for any constant C. Find the solution that satisfies the initial condition $x(0) = 1$. What is the interval of existence of this solution?

We evaluate both sides of (1.19) for $x(s) = 2 - Ce^{-s}$.

$$x'(s) = Ce^{-s}$$
$$2 - x = 2 - (2 - Ce^{-s}) = Ce^{-s}$$

They are the same, so the differential equation is solved for all $s \in (-\infty, \infty)$. In addition,

$$x(0) = 2 - Ce^{-0} = 2 - C.$$

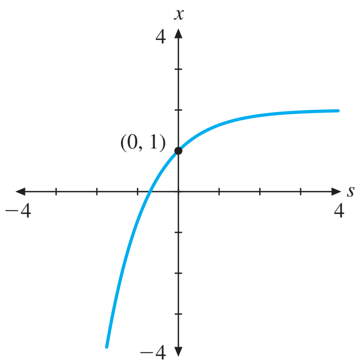


Figure 3. Solution of $x' = 2 - x$, $x(0) = 1$.

To satisfy the initial condition $x(0) = 1$, we must have $2 - C = 1$, or $C = 1$. Therefore, $x(s) = 2 - e^{-s}$ is a solution of the initial value problem. This solution exists for all $s \in (-\infty, \infty)$. Its graph is displayed in Figure 3.

Finally, both $x(s)$ and $x'(s)$ exist and solve the equation on $(-\infty, \infty)$. Therefore, the interval of existence is the whole real line. ●

The geometric meaning of a differential equation and its solutions

Consider the differential equation

$$y' = f(t, y),$$

where the right-hand side $f(t, y)$ is defined for (t, y) in the rectangle

$$R = \{(t, y) \mid a \le t \le b \text{ and } c \le y \le d\}.$$

Let $y(t)$ be a solution of the equation $y' = f(t, y)$, and recall that the graph of the function y is called a solution curve. Because $y(t_0) = y_0$, the point (t_0, y_0) is on the solution curve. The differential equation says that $y'(t_0) = f(t_0, y_0)$. Hence $f(t_0, y_0)$ is the ***slope*** of any solution curve that passes through the point (t_0, y_0).

This interpretation allows us a new, geometric insight into a differential equation. Consider, if you can, a small, slanted line segment with slope $f(t, y)$ attached to every point (t, y) of the rectangle R. The result is called a ***direction field***, because at each (t, y) there is assigned a direction represented by the line with slope $f(t, y)$.

Even for a simple equation like

$$y' = y, \tag{1.20}$$

it is difficult to visualize the direction field. However, a computer can calculate and plot the direction field at a large number of points—a large enough number for us to get a good understanding of the direction field. Each of the standard mathematics programs, Maple, *Mathematica*, and MATLAB®, has the capability to easily produce direction fields. Some hand-held calculators also have this capability. The student will find that the use of computer- or calculator-generated direction fields will greatly assist their understanding of differential equations. A computer-generated direction field for equation (1.20) is given in Figure 4.

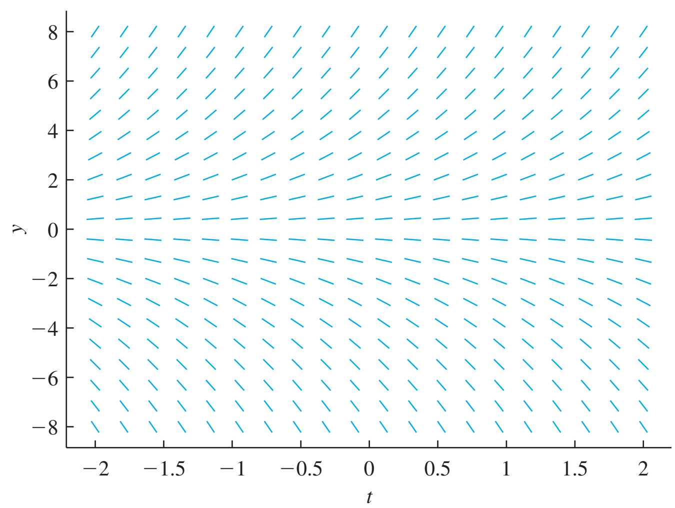


Figure 4. The direction field for $y' = y$.

The direction field is the geometric interpretation of a differential equation. However, the direction field view also gives us a new interpretation of a solution. Associated to the solution $y(t)$, we have the solution curve in the ty-plane. At each point $(t, y(t))$ on the solution curve the curve must have slope equal to $y'(t) = f(t, y(t))$. In other words, the solution curve must be tangent to the direction field at every point. Thus finding a solution to the differential equation is equivalent to the geometric problem of finding a curve in the ty-plane that is tangent to the direction field at every point.

For example, note how the solution curve of

$$y' = y, \quad y(0) = 1 \tag{1.21}$$

in Figure 5 is tangent to the direction field at each point (t, y) on the solution curve.

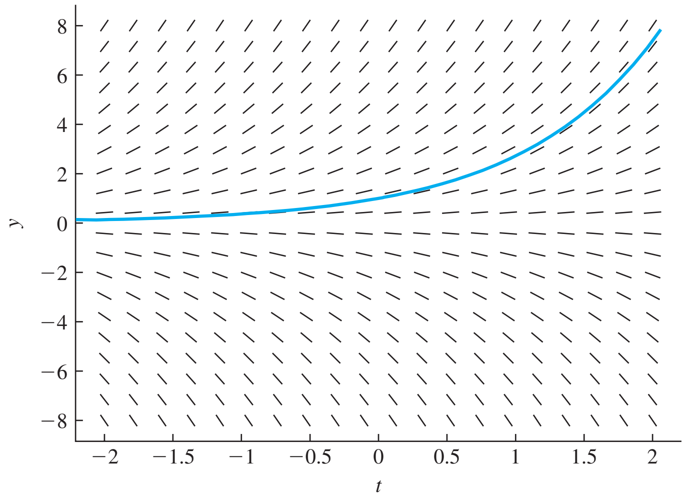


Figure 5. The solution curve is tangent to the direction field.

Approximate numerical solutions

The direction field hints at how we might produce a numerical solution of an initial value problem. To find a solution curve for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, first plot the point $P_0 = (t_0, y_0)$. Because the slope of the solution curve at P_0 is given by $f(t_0, y_0)$, move a prescribed distance along a line with slope $f(t_0, y_0)$ to the point $P_1 = (t_1, y_1)$. Next, because the slope of the solution curve at P_1 is given by $f(t_1, y_1)$, move along a line with slope $f(t_1, y_1)$ to the point $P_2 = (t_2, y_2)$. Continue in this manner to produce an approximate solution curve of the initial value problem.

This technique is used in Figure 6 to produce an approximate solution of equation (1.21) and is the basic idea behind ***Euler's method***, an algorithm used to find numerical solutions of initial value problems. Clearly, if we decrease the distance between consecutively plotted points, we should obtain an even better approximation of the actual solution curve.

Using a numerical solver

We assume that each of our readers has access to a computer. Furthermore, we assume that this computer has software designed to produce numerical solutions of initial value problems. For many purposes a hand-held graphics calculator will suffice. There is a wide variety of software packages available for the study of

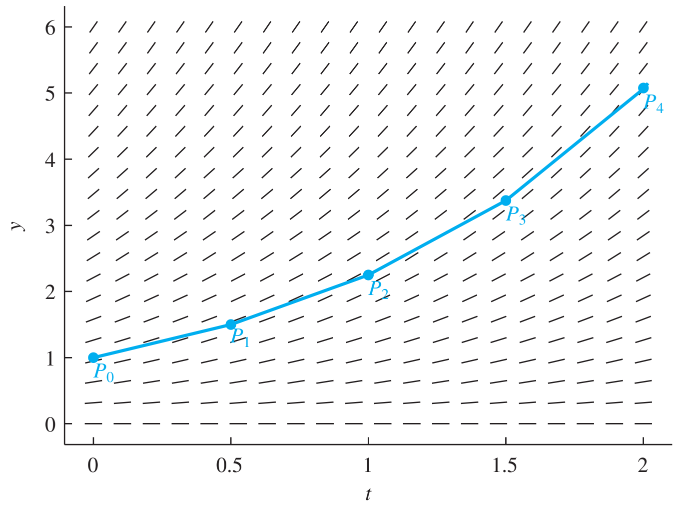


Figure 6. An approximate solution curve of $y' = y$, $y(0) = 1$.

differential equations. Some of these packages are commercial, some are shareware, and some are even freeware. We will assume that you have access to a solver that will

- draw direction fields,
- provide numerical solutions of differential equations and systems of differential equations, and
- plot solutions of differential equations and systems of differential equations.

Test drive your solver

Let's test our solvers in order to assure ourselves that they will provide adequate support for the material in this text.

Example 1.22 Use a numerical solver to compute and plot the solution of the initial value problem

$$y' = y^2 - t, \quad y(4) = 0 \tag{1.23}$$

over the t-interval $[-2, 10]$.

Although solvers differ widely, they do share some common characteristics. First, you need to input the differential equation, and you will probably have to identify the independent variable — in this case t. Most solvers require that you specify a display window, rectangle in which the solution will be drawn. In this case we choose the bounds $-2 \le t \le 10$ and $-4 \le y \le 4$.

Finally, you need to enter the initial condition $y(4) = 0$ and plot the solution. If your solver can superimpose the solution on a direction field, then your plot should look similar to that shown in Figure 7. ●

Qualitative methods

We are unable at this time to find analytic, closed-form solutions to the equation

$$y' = 1 - y^2. \tag{1.24}$$

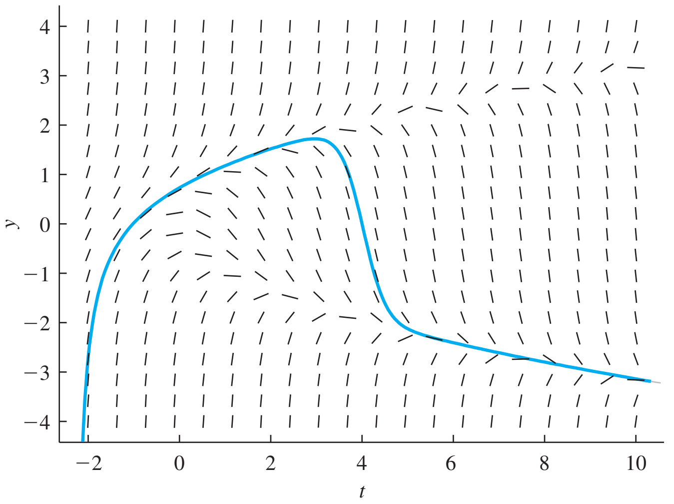


Figure 7. The solution curve for $y' = y^2 - t$, $y(4) = 0$.

This situation will be remedied in the next section. However, the lack of closed-form solutions does not prevent us from using a bit of qualitative mathematical reasoning to investigate a number of important qualities of the solutions of this equation.

Some information about the solutions can be gleaned by looking at the direction field for the equation (1.24) in Figure 8. Notice that the lines $y = 1$ and $y = -1$ seem to be tangent to the direction field. It is easy to verify directly that the constant functions

$$y_1(t) = -1 \quad \text{and} \quad y_2(t) = 1 \tag{1.25}$$

are solutions to equation (1.24).

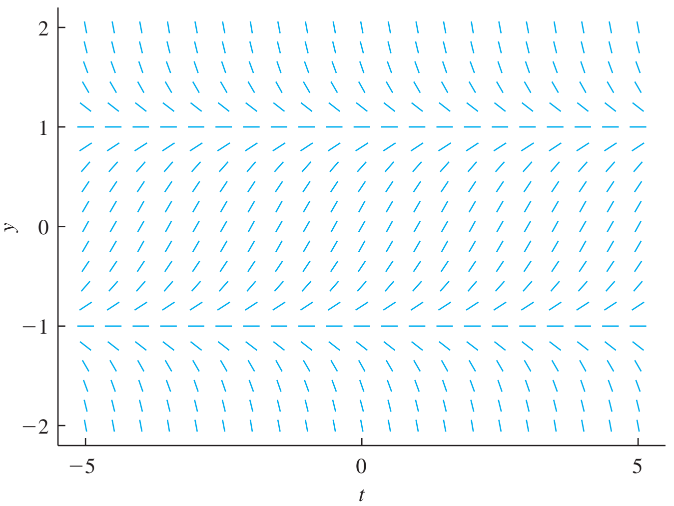


Figure 8. The direction field for the equation $y' = 1 - y^2$.

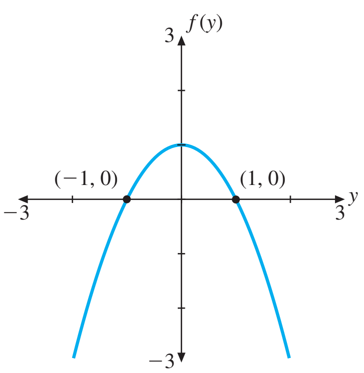


Figure 9. The graph of $f(y) = 1 - y^2$.

To see how we might find such constant solutions, consider the function $f(y) = 1 - y^2$, which is the right-hand side of (1.24). The graph of f is shown in Figure 9. Notice that $f(y) = 0$ only for $y = -1$ and $y = 1$. Each of these points (called ***equilibrium points***) gives rise to one of the solutions we found in (1.25). These

equilibrium solutions are the solutions that can be "seen" in the direction field in Figure 8. They are shown plotted in blue in Figure 10.

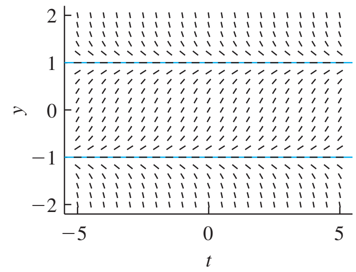


Figure 10. Equilibrium solutions to the equation $y' = 1 - y^2$.

Next we notice that $f(y) = 1 - y^2$ is positive if $-1 < y < 1$ and negative otherwise. Thus, if $y(t)$ is a solution to equation (1.24), and $-1 < y < 1$, then

$$y' = 1 - y^2 > 0.$$

Having a positive derivative, y is an increasing function.

How large can a solution $y(t)$ get? If it gets larger than 1, then $y' = 1 - y^2 < 0$, so $y(t)$ will be decreasing. We cannot complete this line of reasoning at this point, but in Section 2.9 we will develop the argument, and we will be able to conclude that if $y(0) = y_0 > 1$, then $y(t)$ is decreasing and $y(t) \to 1$ as $t \to \infty$.

On the other hand, if $y(0) = y_0$ satisfies $-1 < y_0 < 1$, then $y' = 1 - y^2 > 0$, so $y(t)$ will be increasing. We will again conclude that $y(t)$ increases and approaches 1 as $t \to \infty$. Thus any solution to the equation $y' = 1 - y^2$ with an initial value $y_0 > -1$ approaches 1 as $t \to \infty$.

Finally, if we consider a solution $y(t)$ with $y(0) = y_0 < -1$, then a similar analysis shows that $y'(t) = 1 - y^2 < 0$, so $y(t)$ is decreasing. As $y(t)$ decreases, its derivative $y'(t) = 1 - y^2$ gets more and more negative. Hence, $y(t)$ decreases faster and faster and must approach $-\infty$ as t increases. Typical solutions to equation (1.24) are shown in Figure 11. These solutions were found with a computer, but their qualitative nature can be found simply by looking at the equation.

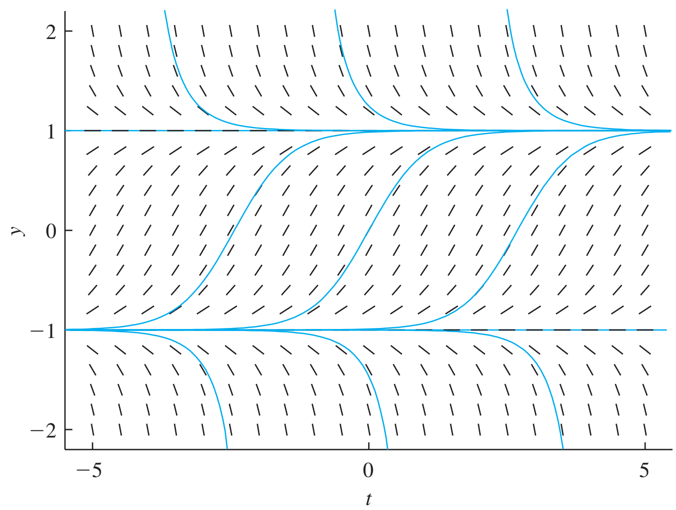


Figure 11. Typical solutions to the equation $y' = 1 - y^2$.

EXERCISES

In Exercises 1 and 2 , given the function ϕ, place the ordinary differential equation $\phi(t, y, y') = 0$ in normal form.

1. $\phi(x, y, z) = x^2 z + (1 + x)y$

2. $\phi(x, y, z) = xz - 2y - x^2$

In Exercises 3–6, show that the given solution is a general solution of the differential equation. Use a computer or calculator to sketch the solutions for the given values of the arbitrary constant. Experiment with different intervals for t until you have a plot that shows what you consider to be the most important behavior of the family.

3. $y' = -ty$, $y(t) = Ce^{-(1/2)t^2}$, $C = -3, -2, \dots, 3$

4. $y' + y = 2t$, $y(t) = 2t - 2 + Ce^{-t}$, $C = -3, -2, \dots, 3$

5. $y' + (1/2)y = 2\cos t$, $y(t) = (4/5)\cos t + (8/5)\sin t + Ce^{-(1/2)t}$, $C = -5, -4, \dots, 5$

6. $y' = y(4 - y)$, $y(t) = 4/(1 + Ce^{-4t})$, $C = 1, 2, \dots, 5$

7. A general solution may fail to produce all solutions of a differential equation. In Exercise 6, show that $y = 0$ is a solution of the differential equation, but no value of C in the given general solution will produce this solution.

8. (a) Use implicit differentiation to show that $t^2 + y^2 = C^2$ implicitly defines solutions of the differential equation $t + yy' = 0$.

(b) Solve $t^2 + y^2 = C^2$ for y in terms of t to provide explicit solutions. Show that these functions are also solutions of $t + yy' = 0$.

(c) Discuss the interval of existence for each of the solutions in part (b).

(d) Sketch the solutions in part (b) for $C = 1, 2, 3, 4$.

9. (a) Use implicit differentiation to show that $t^2 - 4y^2 = C^2$ implicitly defines solutions of the differential equation $t - 4yy' = 0$.

(b) Solve $t^2 - 4y^2 = C^2$ for y in terms of t to provide explicit solutions. Show that these functions are also solutions of $t - 4yy' = 0$.

(c) Discuss the interval of existence for each of the solutions in part (b).

(d) Sketch the solutions in part (b) for $C = 1, 2, 3, 4$.

10. Show that $y(t) = 3/(6t - 11)$ is a solution of $y' = -2y^2$, $y(2) = 3$. Sketch this solution and discuss its interval of existence. Include the initial condition on your sketch.

11. Show that $y(t) = 4/(1 - 5e^{-4t})$ is a solution of the initial value problem $y' = y(4 - y)$, $y(0) = -1$. Sketch this solution and discuss its interval of existence. Include the initial condition on your sketch.

In Exercises 12–15, use the given general solution to find a solution of the differential equation having the given initial condition. Sketch the solution, the initial condition, and discuss the solution's interval of existence.

12. $y' + 4y = \cos t$, $y(t) = (4/17)\cos t + (1/17)\sin t + Ce^{-4t}$, $y(0) = -1$

13. $ty' + y = t^2$, $y(t) = (1/3)t^2 + C/t$, $y(1) = 2$

14. $ty' + (t + 1)y = 2te^{-t}$, $y(t) = e^{-t}(t + C/t)$, $y(1) = 1/e$

15. $y' = y(2 + y)$, $y(t) = 2/(-1 + Ce^{-2t})$, $y(0) = -3$

16. Maple, when asked for the solution of the initial value problem $y' = \sqrt{y}$, $y(0) = 1$, returns two solutions: $y(t) = (1/4)(t + 2)^2$ and $y(t) = (1/4)(t - 2)^2$. Present a thorough discussion of this response, including a check and a graph of each solution, interval of existence, and so on. *Hint:* Remember that $\sqrt{a^2} = |a|$.

In Exercises 17–20, plot the direction field for the differential equation by hand. Do this by drawing short lines of the appropriate slope centered at each of the integer valued coordinates (t, y), where $-2 \le t \le 2$ and $-1 \le y \le 1$.

17. $y' = y + t$

18. $y' = y^2 - t$

19. $y' = t\tan(y/2)$

20. $y' = (t^2 y)/(1 + y^2)$

In Exercises 21–24, use a computer to draw a direction field for the given first-order differential equation. Use the indicated bounds for your display window. Obtain a printout and use a pencil to draw a number of possible solution trajectories on the direction field. If possible, check your solutions with a computer.

21. $y' = -ty$, $R = \{(t, y) : -3 \le t \le 3, -5 \le y \le 5\}$

22. $y' = y^2 - t$, $R = \{(t, y) : -2 \le t \le 10, -4 \le y \le 4\}$

23. $y' = t - y + 1$, $R = \{(t, y) : -6 \le t \le 6, -6 \le y \le 6\}$

24. $y' = (y + t)/(y - t)$, $R = \{(t, y) : -5 \le t \le 5, -5 \le y \le 5\}$

For each of the initial value problems in Exercises 25–28 use a numerical solver to plot the solution curve over the indicated interval. Try different display windows by experimenting with the bounds on y. *Note:* Your solver might require that you first place the differential equation in normal form.

25. $y + y' = 2$, $y(0) = 0$, $t \in [-2, 10]$

26. $y' + ty = t^2$, $y(0) = 3$, $t \in [-4, 4]$

27. $y' - 3y = \sin t$, $y(0) = -3$, $t \in [-6\pi, \pi/4]$

28. $y' + (\cos t)y = \sin t$, $y(0) = 0$, $t \in [-10, 10]$

Some solvers allow the user to choose dependent and independent variables. For example, your solver may allow the equation $r' = -2sr + e^{-s}$, but other solvers will insist that you change variables so that the equation reads $y' = -2ty + e^{-t}$, or $y' = -2xy + e^{-x}$, should your solver require t or x as the independent variable. For each of the initial value problems in Exercises 29 and 30, use your solver to plot solution curves over the indicated interval.

29. $r' + xr = \cos(2x)$, $r(0) = -3$, $x \in [-4, 4]$

30. $T' + T = s$, $T(-3) = 0$, $s \in [-5, 5]$

In Exercises 31–34, plot solution curves for each of the initial conditions on one set of axes. Experiment with the different display windows until you find one that exhibits what you feel is all of the important behavior of your solutions. *Note:* Selecting a good display window is an art, a skill developed with experience. Don't become overly frustrated in these first attempts.

31. $y' = y(3 - y)$, $y(0) = -2, -1, 0, 1, 2, 3, 4, 5$

32. $x' - x^2 = t$, $x(0) = -2, 0, 2$, $x(2) = 0$, $x(4) = -3, 0, 3$, $x(6) = 0$

33. $y' = \sin(xy)$, $y(0) = 0.5, 1.0, 1.5, 2.0, 2.5$

34. $x' = -tx$, $x(0) = -3, -2, -1, 0, 1, 2, 3$

35. Bacteria in a petri dish is growing according to the equation

$$\frac{dP}{dt} = 0.44P,$$

where P is the mass of the accumulated bacteria (measured in milligrams) after t days. Suppose that the initial mass of the bacterial sample is 1.5 mg. Use a numerical solver to estimate the amount of bacteria after 10 days.

36. A certain radioactive substance is decaying according to the equation

$$\frac{dA}{dt} = -0.25A,$$

where A is the amount of substance in milligrams remaining after t days. Suppose that the initial amount of the substance present is 400 mg. Use a numerical solver to estimate the amount of substance remaining after 4 days.

37. The concentration of pollutant in a lake is given by the equation

$$\frac{dc}{dt} = -0.055c,$$

where c is the concentration of the pollutant at t days. Suppose that the initial concentration of pollutant is 0.10. A concentration level of $c = 0.02$ is deemed safe for the fish population in the lake. If the concentration varies according to the model, how long will it be before the concentration reaches a level that is safe for the fish population?

38. An aluminum rod is heated to a temperature of 300°C. Suppose that the rate at which the rod cools is proportional to the difference between the temperature of the rod and the temperature of the surrounding air (20°C). Assume a proportionality constant $k = 0.085$ and time is measured in minutes. How long will it take the rod to cool to 100°C?

39. You're told that the "carrying capacity" for an environment populated by "critters" is 100. Further, you're also told that the rate at which the critter population is changing is proportional to the product of the number of critters and the number of critters less than the carrying capacity. Assuming a constant of proportionality $k = 0.00125$ and an initial critter population of 20, use a numerical solver to determine the size of the critter population after 30 days.

2.2 Solutions to Separable Equations

An unstable nucleus is radioactive. At any instant, it can emit a particle, transforming itself into a different nucleus in the process. For example, ^{238}U is an alpha emitter that decays spontaneously according to the scheme $^{238}U \to {}^{234}Th + {}^{4}He$, where ^{4}He is the alpha particle. In a sample of ^{238}U, a certain percentage of the nuclei will decay during a given observation period. If at time t the sample contains $N(t)$ radioactive nuclei, then we expect that the number of nuclei that decay in the time interval Δt will be approximately proportional to both N and Δt. In symbols,

$$\Delta N = N(t + \Delta t) - N(t) \approx -\lambda N(t)\Delta t, \tag{2.1}$$

where $\lambda > 0$ is a constant of proportionality. The minus sign is indicative of the fact that there are fewer radioactive nuclei at time $t + \Delta t$ than there are at time t.

Dividing both sides of equation (2.1) by Δt, then taking the limit as $\Delta t \to 0$,

$$N'(t) = \lim_{\Delta t \to 0} \frac{N(t + \Delta t) - N(t)}{\Delta t} = -\lambda N(t).$$

This equation is one that arises often in applications. Because of the form of its solutions, the equation

$$N' = -\lambda N \tag{2.2}$$

is called the ***exponential equation***.

Equation (2.2) is an example of what is called a ***separable equation*** because it can be rewritten with its variables separated and then easily solved. To do this, we first write the equation using dN/dt instead of N',

$$\frac{dN}{dt} = -\lambda N. \tag{2.3}$$

Next, we separate the variables by putting every expression involving the unknown function N on the left and everything involving the independent variable t on the right. This includes dN and dt. The result is

$$\frac{1}{N}\, dN = -\lambda\, dt. \tag{2.4}$$

It is important to note that this step is valid only if $N \neq 0$, since we cannot divide by zero. Then we integrate both sides of equation (2.4), getting[2]

$$\int \frac{1}{N}\,dN = -\lambda \int dt, \quad \text{or}$$
$$\ln|N| = -\lambda t + C. \tag{2.5}$$

It remains to solve for N. Taking the exponential of both sides of equation (2.5), we get

$$|N(t)| = e^{-\lambda t + C} = e^C e^{-\lambda t}. \tag{2.6}$$

Since e^C and $e^{-\lambda t}$ are both positive, there are two cases

$$N(t) = \begin{cases} e^C e^{-\lambda t}, & \text{if } N > 0; \\ -e^C e^{-\lambda t}, & \text{if } N < 0. \end{cases}$$

We can simplify the solution by introducing

$$A = \begin{cases} e^C, & \text{if } N > 0; \\ -e^C, & \text{if } N < 0. \end{cases}$$

Therefore, the solution is also described by the simpler formula

$$N(t) = Ae^{-\lambda t}, \tag{2.7}$$

where A is a constant different from zero, but otherwise arbitrary.

In arriving at equation (2.4), we divided both sides of equation (2.3) by N, and this procedure is not valid when $N = 0$. We will discuss this a bit later. For now, let's notice that if we set $A = 0$ in equation (2.7), we get the constant function $N(t) = 0$, and we can verify by substitution that this is a solution of the original equation, $N' = -\lambda N$. Consequently, equation (2.7) with A completely arbitrary, gives us the solution in all cases.

Example 2.8 ^{32}P, an isotope of phosphorus, is used in leukemia therapy. After 10 hours, 615 mg of an initial 1000 mg sample remain. The ***half-life*** of a radioactive substance is the amount of time required for 50% of the substance to decay. Determine the half-life of ^{32}P.

The differential equation $N' = -\lambda N$ was used to model the number of remaining nuclei. However, the number of nuclei is proportional to the mass, so we will let N represent the mass of the remaining nuclei in this example. As seen earlier, this differential equation has solution

$$N = Ae^{-\lambda t}, \tag{2.9}$$

where A is an arbitrary constant. At time $t = 0$ we have $N = 1000$ mg of the isotope. Substituting these quantities in equation (2.9),

$$1000 = Ae^{-\lambda(0)} = A. \tag{2.10}$$

[2] Our understanding of integration first has us use two constants of integration,

$$\ln|N| + C_1 = -\lambda t + C_2.$$

We get (2.5) by setting $C = C_2 - C_1$. This combining of the two constants into one works in the solution of any separable equation.

Consequently, equation (2.9) becomes

$$N = 1000e^{-\lambda t}. \tag{2.11}$$

After $t = 10\,\text{hr}$, only $N = 615\,\text{mg}$ of the substance remains. Substituting these values into equation (2.11), we get

$$615 = 1000e^{-\lambda(10)}. \tag{2.12}$$

Using a little algebra and a calculator to compute a logarithm shows that $\lambda = 0.04861$, correct to six decimal places, and equation (2.11) becomes

$$N = 1000e^{-0.04861t}. \tag{2.13}$$

To find the half-life, we substitute $N = 500\,\text{mg}$ in equation (2.13).

$$500 = 1000e^{-0.04861t}$$

Solving for t, we find that the half-life of the isotope is approximately 14.3 hours. ●

A large number of equations are separable and can be solved exactly like we solved the exponential equation. Let's look at another example.

Example 2.14 Solve the differential equation

$$y' = ty^2. \tag{2.15}$$

Again, we rewrite the equation using dy/dt instead of y', so

$$\frac{dy}{dt} = ty^2. \tag{2.16}$$

Next we separate the variables by putting every expression involving the unknown function y on the left and everything involving the independent variable t on the right, including dy and dt. The result is

$$\frac{1}{y^2}\,dy = t\,dt. \tag{2.17}$$

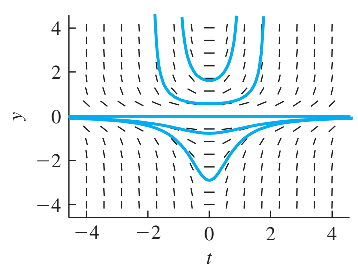


Figure 1. Several solutions to $y' = ty^2$.

Notice that this step is valid only if $y \neq 0$, since we cannot divide by zero. Next we integrate both sides of equation (2.17), getting

$$\int \frac{1}{y^2}\,dy = \int t\,dt, \quad \text{or} \quad -\frac{1}{y} = \frac{1}{2}t^2 + C. \tag{2.18}$$

Finally, we solve equation (2.18) for y. The equation for the solution is

$$y(t) = \frac{-1}{\frac{1}{2}t^2 + C} = \frac{-2}{t^2 + 2C}. \tag{2.19}$$

Several solutions are shown in Figure 1. Included among the functions plotted in Figure 1 is the constant function $y(t) = 0$. It is easily verified by substitution that this is a solution of (2.15), although no finite value of C in equation (2.19) will yield this solution. We will have more to say about this on page 30. ●

Treating dy and dt as mathematical entities, as we did in separating the variables in equation (2.17), may be troublesome to you. If so, it is probably because you have learned your calculus very well. We will explain this step at the end of this section under the heading "Why separation of variables works."

The general method

Clearly the key step in this method is the separation of variables. This is the step going from equation (2.3) to equation (2.4) or from equation (2.16) to equation (2.17). The method of solution illustrated here will work whenever we can perform this step, and this can be done for any equation of the two equivalent forms

$$\frac{dy}{dt} = \frac{g(t)}{h(y)} \tag{2.20}$$

and

$$\frac{dy}{dt} = g(t)f(y). \tag{2.21}$$

Equations of either form are called ***separable*** differential equations. For both we can separate the variables.

The method we used to solve equation (2.16) will work for any separable equation.

> We can solve any separable equation of the form (2.21) using the following three steps.
>
> **1.** Separate the variables: $\dfrac{dy}{f(y)} = g(t)\,dt.$
>
> **2.** Integrate both sides: $\displaystyle\int \frac{dy}{f(y)} = \int g(t)\,dt.$
>
> **3.** Solve for the solution $y(t)$, if possible.

Avoiding division by zero

When separating the variables we do have to worry about dividing by zero, but otherwise things work well. What about those points where $f(y) = 0$ in equation (2.21)? It turns out to be quite easy to find the solutions in such a case, since if $f(y_0) = 0$, then by substitution we see that the constant function $y(t) = y_0$ is a solution to (2.21).

In particular, the function $y(t) = 0$ is a solution to the equation $y' = ty^2$. We found in (2.19) that, under the assumption that $y \neq 0$, the general solution to the equation $y' = ty^2$ is

$$y(t) = \frac{-2}{t^2 + 2C}.$$

If we naively substitute the initial condition $y(0) = 0$ into this general solution, we get $0 = -1/C$. No finite value of the constant C solves this equation. This should not be a surprise, since (2.19) was derived on the assumption that $y \neq 0$. Nevertheless, we will want to call (2.19) a general solution to equation (2.16). We define a ***general solution*** to a differential equation to be a family of solutions depending on sufficiently many parameters to give all but finitely many solutions.

Thus the general solution to a differential equation does not always yield the solution to every initial value problem, and for separable equations this is related to the problem of dividing by 0. In the case of $y' = ty^2$, we can find the exceptional solution by setting $C = \infty$. This often the case, but we will not explore this further.

Using definite integration

Sometimes it is useful to use definite integrals when solving initial value problems for separable equations.

Example 2.22 A can of beer at 40°F is placed into a room where the temperature is 70°F. After 10 minutes the temperature of the beer is 50°F. What is the temperature of the beer as a function of time? What is the temperature of the beer 30 minutes after the beer was placed into the room?

According to ***Newton's law of cooling***, the rate of change of an object's temperature (T) is proportional to the difference between its temperature and the ambient temperature (A). Thus we have

$$\frac{dT}{dt} = -k(T - A). \tag{2.23}$$

We introduce the minus sign so that the proportionality constant k is positive. Notice that if $T < A$, the temperature of the object will be increasing. The equation is separable, so we separate variables to get

$$\frac{dT}{T - A} = -k\,dt.$$

The next step is to integrate both sides, but this time let's use definite integrals to bring in the initial condition $T(0) = T_0$. Since $t = 0$ corresponds to $T = T_0$, we have

$$\int_{T_0}^{T} \frac{ds}{s - A} = -k \int_0^t du.$$

Notice that we changed the variables of integration because we want the upper limits of our integrals to be T and t. Performing the integration, we get

$$\ln \frac{|T - A|}{|T_0 - A|} = \ln|T - A| - \ln|T_0 - A| = -kt.$$

We can solve for T by exponentiating, and since $T - A$ and $T_0 - A$ both have the same sign, our answer is

$$T(t) = A + (T_0 - A)e^{-kt}. \tag{2.24}$$

We first use the fact that $T(10) = 50$ in addition to the initial condition $T(0) = T_0 = 40$ and the ambient temperature $A = 70$ to evaluate k. Equation (2.24) becomes $50 = 70 - 30e^{-10k}$. Therefore, $k = \ln(3/2)/10 = 0.0405$. Thus from equation (2.24) we see that the temperature is

$$T(t) = 70 - 30e^{-0.0405t}.$$

After 30 minutes the temperature is 61.1°F. The solution is plotted in Figure 2. ●

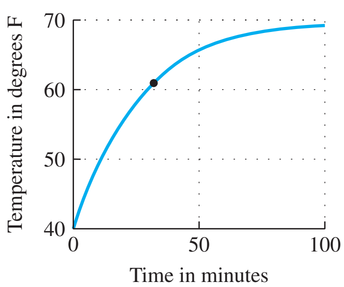


Figure 2. The temperature of the can of beer in Example 2.22.

Implicitly defined solutions

After the integration step, we need to solve for the solution. However, this is not always easy. In fact, it is not always possible. We will look at a series of examples.

Example 2.25 Find the solutions of the equation $y' = e^x/(1+y)$, having initial conditions $y(0) = 1$ and $y(0) = -4$.

Separate the variables and integrate.

$$(1 + y)\,dy = e^x\,dx$$
$$y + \frac{1}{2}y^2 = e^x + C \tag{2.26}$$

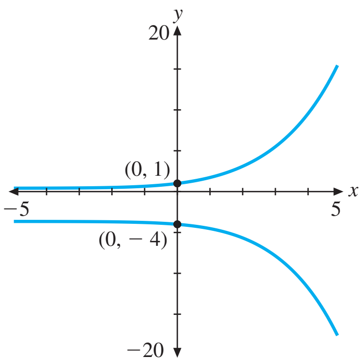


Figure 3. $y = -1 + \sqrt{2 + 2e^x}$ passes through (0, 1), while $y = -1 - \sqrt{7 + 2e^x}$ passes through (0, −4).

Rearrange equation (2.26) as

$$y^2 + 2y - 2(e^x + C) = 0. \tag{2.27}$$

This is an implicit equation for $y(x)$ that we can solve using the quadratic formula.

$$y(x) = \frac{1}{2}\left[-2 \pm \sqrt{4 + 8(e^x + C)}\right]$$
$$= -1 \pm \sqrt{1 + 2(e^x + C)}$$

We get two solutions from the quadratic formula, and the initial condition will dictate which solution we choose. If $y(0) = 1$, then we must use the positive square root and we find that $C = 1/2$. The solution is

$$y(x) = -1 + \sqrt{2 + 2e^x}. \tag{2.28}$$

On the other hand, if $y(0) = -4$, then we must use the negative square root and we find that $C = 3$. The solution in this case is

$$y(x) = -1 - \sqrt{7 + 2e^x}. \tag{2.29}$$

Both solutions are shown in Figure 3.

What about the interval of existence? A quick glance reveals that each solution is defined on the interval $(-\infty, \infty)$. Some calculation will reveal that $y'(x)$ is also defined on $(-\infty, \infty)$. However, for each solution to satisfy the equation $y' = e^x/(1 + y)$, y must not equal -1. Fortunately, neither solution (2.28) or (2.29) can ever equal -1. Therefore, the interval of existence is $(-\infty, \infty)$. ●

Let's be sure we know what the terminology means. An ***explicit*** solution is one for which we have a formula as a function of the independent variable. For example, (2.28) is an explicit solution to the equation in Example 2.25. In contrast, (2.27) is an implicit equation for the solution. In this example, the implicit equation can be solved easily to find an explicit equation, but this is not always the case.

Unfortunately, implicit solutions occur frequently. Consider again the general problem in the form $dy/dt = g(t)/h(y)$. Separating variables and integrating, we get

$$\int h(y)\,dy = \int g(t)\,dt. \tag{2.30}$$

If we let

$$H(y) = \int h(y)\,dy \quad \text{and} \quad G(t) = \int g(t)\,dt,$$

and then introduce a constant of integration, equation (2.30) can be rewritten as

$$H(y) = G(t) + C. \tag{2.31}$$

Unless $H(y) = y$, and therefore $h(y) = 1$, this is an implicit equation for $y(t)$. To find an explicit solution we must be able to compute the inverse function H^{-1}. If this is possible, then we have

$$y(t) = H^{-1}(G(t) + C).$$

Let's do one more example.

Example 2.32 Find the solutions to the differential equation

$$x' = \frac{2tx}{1+x},$$

having initial conditions $x(0) = 1$, $x(0) = -2$, and $x(0) = 0$.

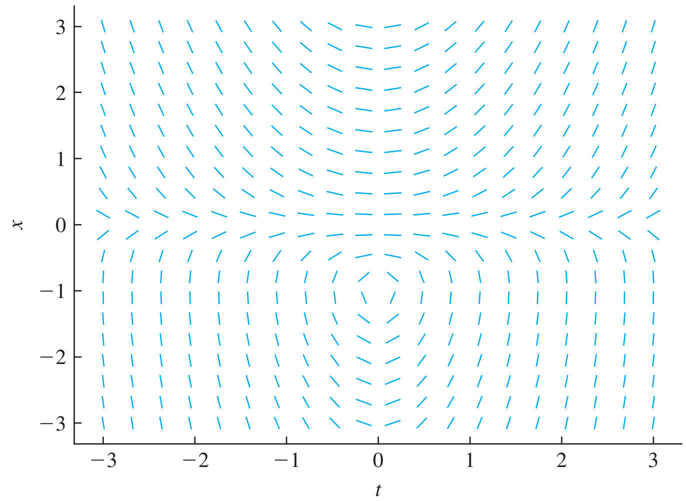


Figure 4. The direction field for $x' = 2tx(1+x)$.

The direction field for this equation is shown in Figure 4. This equation is separable since it can be written as

$$\frac{dx}{dt} = 2t\frac{x}{1+x}.$$

When we separate variables, we get

$$\left(1+\frac{1}{x}\right) dx = 2t\,dt,$$

assuming that $x \neq 0$. Integrating, we get

$$x + \ln(|x|) = t^2 + C, \tag{2.33}$$

where C is an arbitrary constant. For the initial condition $x(0) = 1$, this becomes $1 = C$. Hence our solution is implicitly defined by $x+\ln(|x|) = t^2+1$. The function $\ln(|x|)$ is not defined at $x = 0$, so our solution can never be equal to 0. Since our initial condition is positive, and a solution must be continuous, our solution $x(t)$ must be positive for all t. Hence $|x| = x$ and our solution is given implicitly by

$$x + \ln(x) = t^2 + 1. \tag{2.34}$$

This is as far as we can go. We cannot solve equation (2.34) explicitly for $x(t)$, so we have to be satisfied with this as our answer. The solution x is defined implicitly by equation (2.34).

For the initial condition $x(0) = -2$, we can find the constant C in the same manner. We get $-2 + \ln(|-2|) = C$, or $C = \ln 2 - 2$. Hence the solution is defined implicitly by

$$x + \ln(|x|) = t^2 + \ln 2 - 2.$$

This time our initial condition is negative, so $|x| = -x$, and our implicit equation for the solution is

$$x + \ln(-x) = t^2 + \ln 2 - 2.$$

For the initial condition $x(0) = 0$, we cannot divide by $x/(1 + x)$ to separate variables. However, we know that this means that $x(t) = 0$ is a solution. We can easily verify that by direct substitution. Thus we do get an explicit formula for the solution with this initial condition. ●

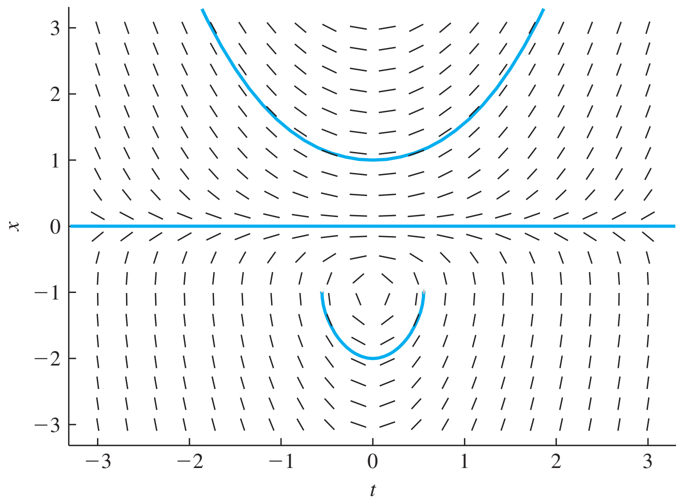


Figure 5. Solutions to $x' = 2tx(1 + x)$.

The solutions sought in the previous example were computed numerically and are plotted in Figure 5. Since the solutions are defined implicitly, it is a difficult task to visualize them without the aid of numerical methods.

Why separation of variables works

If we start with a separable equation

$$y' = g(t)\big/h(y), \tag{2.35}$$

then separation of variables leads to the equation

$$h(y)\,dy = g(t)\,dt. \tag{2.36}$$

However, many readers will have been taught that the terms dy and dt have no meaning and so equation (2.36) has no meaning. Yet the method works, so what is going on here?

To understand this better, let's start with (2.35) and perform legitimate steps

$$y' = g(t)/h(y) \quad \text{or} \quad h(y)y' = g(t).$$

Integrating both sides with respect to t, we get

$$\int h(y(t))y'(t)\,dt = \int g(t)\,dt.$$

The integral on the left contains the expression $y'(t)\,dt$. This is inviting us to change the variable of integration to y, since when we do that, we use the equation $dy = y'(t)\,dt$. Making the change of variables leads to

$$\int h(y)\,dy = \int g(t)\,dt. \tag{2.37}$$

Notice the similarity between (2.36) and (2.37). Equation (2.36), which has no meaning by itself, acquires a precise meaning when both sides are integrated. Since this is precisely the next step that we take when solving separable equations, we can be sure that our method is valid.

We mention in closing that the objects in (2.36), $h(y)\,dy$ and $g(t)\,dt$, can be given meaning as formal objects that can be integrated. They are called ***differential forms***, and the special cases like dy and dt are called ***differentials***. The basic formula connecting differentials dy and dt when y is a function of t is

$$dy = y'(t)\,dt,$$

the change-of-variables formula in integration. These techniques will assume greater importance in Section 2.6, where we will deal with exact equations. The use of differential forms is very important in the study of the calculus of functions of several variables and especially in applications to geometry and to parts of physics.

EXERCISES

In Exercises 1–12, find the general solution of the indicated differential equation. If possible, find an explicit solution.

1. $y' = xy$

2. $xy' = 2y$

3. $y' = e^{x-y}$

4. $y' = (1 + y^2)e^x$

5. $y' = xy + y$

6. $y' = ye^x - 2e^x + y - 2$

7. $y' = x/(y + 2)$

8. $y' = xy/(x - 1)$

9. $x^2y' = y \ln y - y'$

10. $xy' - y = 2x^2y$

11. $y^3y' = x + 2y'$

12. $y' = (2xy+2x)/(x^2-1)$

In Exercises 13–18, find the exact solution of the initial value problem. Indicate the interval of existence.

13. $y' = y/x,\ y(1) = -2$

14. $y' = -2t(1 + y^2)/y,\ y(0) = 1$

15. $y' = (\sin x)/y,\ y(\pi/2) = 1$

16. $y' = e^{x+y},\ y(0) = 0$

17. $y' = (1 + y^2),\ y(0) = 1$

18. $y' = x/(1 + 2y),\ y(-1) = 0$

In Exercises 19–22, find exact solutions for each given initial condition. State the interval of existence in each case. Plot each exact solution on the interval of existence. Use a numerical solver to duplicate the solution curve for each initial value problem.

19. $y' = x/y,\ y(0) = 1,\ y(0) = -1$

20. $y' = -x/y,\ y(0) = 2,\ y(0) = -2$

21. $y' = 2 - y,\ y(0) = 3,\ y(0) = 1$

22. $y' = (y^2 + 1)/y,\ y(1) = 2$

23. Suppose that a radioactive substance decays according to the model $N' = N_0e^{-\lambda t}$. Show that the half-life of the radioactive substance is given by the equation

$$T_{1/2} = \frac{\ln 2}{\lambda}. \tag{2.38}$$

24. The half-life of ^{238}U is 4.47×10^7 yr.

(a) Use equation (2.38) to compute the ***decay constant*** λ for ^{238}U.

(b) Suppose that 1000 mg of ^{238}U are present initially. Use the equation $N = N_0e^{-\lambda t}$ and the decay constant determined in part (a) to determine the time for this sample to decay to 100 mg.

25. Tritium, ^3H, is an isotope of hydrogen that is sometimes used as a biochemical tracer. Suppose that 100 mg of ^3H decays to 80 mg in 4 hours. Determine the half-life of ^3H.

26. The isotope Technetium 99m is used in medical imaging. It has a half-life of about 6 hours, a useful feature for radioisotopes that are injected into humans. The Technetium, having such a short half-life, is created artificially on scene by harvesting from a more stable isotope, ^{99}Mb. If 10 g of ^{99m}Tc are "harvested" from the Molybdenum, how much of this sample remains after 9 hours?

27. The isotope Iodine 131 is used to destroy tissue in an overactive thyroid gland. It has a half-life of 8.04 days. If a hospital receives a shipment of 500 mg of ^{131}I, how much of the isotope will be left after 20 days?

28. A substance contains two Radon isotopes, ^{210}Rn [$t_{1/2} = 2.42$ h] and ^{211}Rn [$t_{1/2} = 15$ h]. At first, 20% of the decays come from ^{211}Rn. How long must one wait until 80% do so?

29. Suppose that a radioactive substance decays according to the model $N = N_0 e^{-\lambda t}$.

(a) Show that after a period of $T_\lambda = 1/\lambda$, the material has decreased to e^{-1} of its original value. T_λ is called the ***time constant*** and it is defined by this property.

(b) A certain radioactive substance has a half-life of 12 hours. Compute the time constant for this substance.

(c) If there are originally 1000 mg of this radioactive substance present, plot the amount of substance remaining over four time periods T_λ.

In the laboratory, a more useful measurement is the decay rate R, usually measured in disintegrations per second, counts per minute, etc. Thus, the ***decay rate*** is defined as $R = -dN/dt$. Using the equation $dN/dt = -\lambda N$, it is easily seen that $R = \lambda N$. Furthermore, differentiating the solution $N = N_0 e^{-\lambda t}$ with respect to t reveals that

$$R = R_0 e^{-\lambda t}, \qquad (2.39)$$

in which R_0 is the decay rate at $t = 0$. That is, because R and N are proportional, they both decrease with time according to the same exponential law. Use this idea to help solve Exercises 30–31.

30. Jim, working with a sample of ^{131}I in the lab, measures the decay rate at the end of each day.

Time (days)	Counts (counts/day)	Time (days)	Counts (counts/day)
1	938	6	587
2	822	7	536
3	753	8	494
4	738	9	455
5	647	10	429

Like any modern scientist, Jim wants to use all of the data instead of only two points to estimate the constants R_0 and λ in equation (2.39). He will use the technique of ***regression*** to do so. Use the first method in the following list that your technology makes available to you to estimate λ (and R_0 at the same time). Use this estimate to approximate the half-life of ^{131}I.

(a) Some modern calculators and the spreadsheet Excel can do an exponential regression to directly estimate R_0 and λ.

(b) Taking the natural logarithm of both sides of equation (2.39) produces the result

$$\ln R = -\lambda t + \ln R_0.$$

Now $\ln R$ is a linear function of t. Most calculators, numerical software such as MATLAB®, and computer algebra systems such as *Mathematica* and Maple will do a linear regression, enabling you to estimate $\ln R_0$ and λ (e.g., use the MATLAB® command `polyfit`).

(c) If all else fails, plotting the natural logarithm of the decay rates versus the time will produce a curve that is almost linear. Draw the straight line that in your estimation provides the best fit. The slope of this line provides an estimate of $-\lambda$.

31. A 1.0 g sample of Radium 226 is measured to have a decay rate of 3.7×10^{10} disintegrations/s. What is the half-life of ^{226}Ra in years? *Note:* A chemical constant, called Avogadro's number, says that there are 6.02×10^{23} atoms per mole, a common unit of measurement in chemistry. Furthermore, the atomic mass of ^{226}Ra is 226 g/mol.

32. Radiocarbon dating. Carbon 14 is produced naturally in the earth's atmosphere through the interaction of cosmic rays and Nitrogen 14. A neutron comes along and strikes a ^{14}N nucleus, knocking off a proton and creating a ^{14}C atom. This atom now has an affinity for oxygen and quickly oxidizes as a $^{14}CO_2$ molecule, which has many of the same chemical properties as regular CO_2. Through photosynthesis, the $^{14}CO_2$ molecules work their way into the plant system, and from there into the food chain. The ratio of ^{14}C to regular carbon in living things is the same as the ratio of these carbon atoms in the earth's atmosphere, which is fairly constant, being in a state of equilibrium. When a living being dies, it no longer ingests ^{14}C and the existing ^{14}C in the now defunct life form begins to decay. In 1949, Willard F. Libby and his associates at the University of Chicago measured the half-life of this decay at 5568 ± 30 years, which to this day is known as the ***Libby half-life***. We now know that the half-life is closer to 5730 years, called the ***Cambridge half-life***, but radiocarbon dating labs still use the Libby half-life for technical and historical reasons. Libby was awarded the Nobel prize in chemistry for his discovery.

(a) Carbon 14 dating is a useful dating tool for organisms that lived during a specific time period. Why is that? Estimate this period.

(b) Suppose that the ratio of ^{14}C to carbon in the charcoal on a cave wall is 0.617 times a similar ratio in living wood in the area. Use the Libby half-life to estimate the age of the charcoal.

33. A murder victim is discovered at midnight and the temperature of the body is recorded at 31°C. One hour later, the temperature of the body is 29°C. Assume that the surrounding air temperature remains constant at 21°C. Use Newton's law of cooling to calculate the victim's time of death. *Note:* The "normal" temperature of a living human being is approximately 37°C.

34. Suppose a cold beer at 40°F is placed into a warm room at 70°F. Suppose 10 minutes later, the temperature of the beer is 48°F. Use Newton's law of cooling to find the temperature 25 minutes after the beer was placed into the room.

35. Referring to the previous problem, suppose a 50° bottle of beer is discovered on a kitchen counter in a 70° room. Ten minutes later, the bottle is 60°. If the refrigerator is kept at 40° how long had the bottle of beer been sitting on the counter when it was first discovered?

36. Consider the equation

$$y' = f(at + by + c),$$

where a, b, and c are constants. Show that the substitution $x = at + by + c$ changes the equation to the separable equation $x' = a + bf(x)$. Use this method to find the general solution of the equation $y' = (y + t)^2$.

37. Suppose a curve, $y = y(x)$ lies in the first quadrant and suppose that for each x the piece of the tangent line at $(x, y(x))$ which lies in the first quadrant is bisected by the point $(x, y(x))$. Find $y(x)$.

38. Suppose the projection of the part of the normal line to the graph of $y = y(x)$ from the point $(x, y(x))$ to the x-axis has length 2. Find $y(x)$.

39. Suppose a polar graph $r = r(\theta)$ has the property that θ always equals twice the angle from the radial line (i.e. the line from the origin to $(\theta, r(\theta))$) to the tangent. Find the function $r(\theta)$.

40. Suppose $y(x)$ is a continuous, nonnegative function with $y(0) = 0$. Find $y(x)$ if the area under the curve, $y = y(t)$, from 0 to x is always equal to one-fourth the area of the rectangle with vertices at $(0, 0)$ and $(x, y(x))$.

41. A football, in the shape of an ellipsoid, is lying on the ground in the rain. Its length is 8 inches and its cross section at its widest point is a circular disc of radius 2 inches. A rain drop hits the top half of the football. Find the path that it follows as it runs down the top half of the football. *Hint:* Recall that the gradient of a function $f(x, y)$ points in the (x, y)-direction of maximum increase of f.

42. From Torricelli's law, water in an open tank will flow out through a hole in the bottom at a speed equal to that it would acquire in a free-fall from the level of the water to the hole. A parabolic bowl has the shape of $y = x^2$, $0 \le x \le 1$, (units are feet) revolved around the y-axis. This bowl is initially full of water and at $t = 0$, a hole of radius a is punched at the bottom. How long will it take for the bowl to drain? *Hint:* An object dropped from height h will hit the ground at a speed of $v = \sqrt{2gh}$, where g is the gravitational constant. This formula is derived from equating the kinetic energy on impact, $(1/2)mv^2$, with the work required to raise the object, mgh.

43. Referring to the previous problem, find the shape of the bowl if the water level drops at a constant rate.

44. A destroyer is hunting a submarine in a dense fog. The fog lifts for a moment, disclosing that the submarine lies on the surface 4 miles away. The submarine immediately descends and departs in a straight line in an unknown direction. The speed of the destroyer is three times that of the submarine. What path should the destroyer follow to be certain of intercepting the submarine? *Hint:* Establish a polar coordinate system with the origin located at the point where the submarine was sighted. Look up the formula for arc length in polar coordinates.

2.3 Models of Motion

One of the most intensively studied scientific problems is the study of motion. This is true in particular for the motion of the planets. The history of the ideas involved is one of the most interesting chapters of human history. We will start by giving a brief summary of the development of models of motion.

A brief history of models of motion

The study of the stars and planets is as old as humankind. Even the most primitive people have been fascinated by the nightly display of the stars, and they soon noticed that some objects, now called planets, moved against the background of the "fixed" stars. The systematic study of planetary motion goes back at least 3000 years to the Babylonians, who made the first recorded observations.

Their interest was furthered by the Greek civilization. There were a number of explanations posed, including that of Aristarchus who put the sun at the center of the universe. However, the one that lasted was developed over hundreds of years and culminated with the work of Hipparchus and Ptolemy. It was published in Claudius Ptolemy's *Almagest* in the second century A.D. Their theory was a descriptive model of the motion of the planets. They assumed that the earth was the center of the universe and that everything revolved around it. At first they thought that the planets, the sun, and the moon moved with constant velocities in circular paths around the earth. As they grew more proficient in their measurements they realized that this was not true. They modified their theory by inventing ***epicycles***. These were smaller circles, the centers of which moved with constant velocity along circular paths centered at the earth. The planets moved with constant velocity along

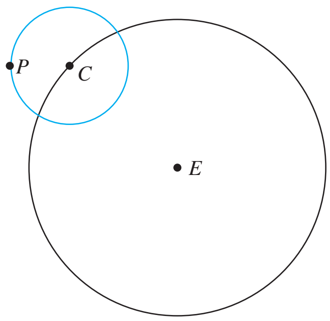


Figure 1. A planet moving on an epicycle.

the epicycles as the epicycles moved around the earth, as illustrated in Figure 1. When this theory proved to be inadequate in some cases, the Greeks added epicycles to the epicycles.

The theory of epicycles enabled the Greeks to compute and predict the motion of the planets. In many ways it was a highly satisfactory scientific theory, but it left many questions unanswered. Most important, why do the planets move in the complicated manner suggested by the theory of epicycles? The explanation was not causal. It was only descriptive in nature.

A major improvement on this theory came in 1543, when Nicholas Copernicus made the radical suggestion that the earth was not the center of the universe. Instead, he proposed that the sun was the center. Of course this required a major change in the thinking of all humankind in matters of religion and philosophy as well as in astronomy. It did, however, make the theory of epicycles somewhat easier, because fewer epicycles were needed to explain the motion of the planets.

Starting in 1609, and based on extensive and careful astronomical observations made by Tycho Brahe, Johannes Kepler postulated three laws of planetary motion:

1. Each planet moves in an ellipse with the sun at one focus.
2. The line between the sun and a planet sweeps out equal areas in equal times.
3. The squares of the periods of revolution of the planets are proportional to the cubes of the semimajor axis of their elliptic orbits.

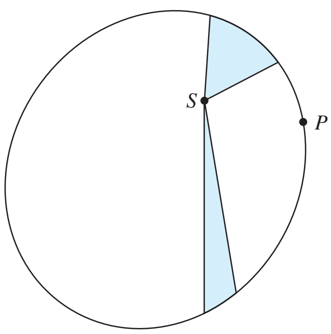


Figure 2. Kepler's second law.

Each of Kepler's laws made a major break with the past. The first abandoned circular motion and the need for epicycles. The second abandoned the uniformity of speed that had been part of the Ptolemaic theory, and replaced it by a beautiful mathematical expression of how fast a planet moved. The first two laws are illustrated in Figure 2. The planet P moves along an ellipse with the sun S at one focus. The two pie-shaped regions have equal area, so the planet will traverse the two arcs in equal times. The third law was equally dramatic, since it displays a commonality in the motion of all of the planets. Although a major accomplishment, Kepler's results remained descriptive. His three laws provided no causal explanation for the motion of the planets.

A causal explanation was provided by Isaac Newton. However he did much more. He made three major advances.[3] First, he proved the fundamental theorem of calculus, and for that reason he is given credit for inventing the calculus. The fundamental theorem made possible the easy evaluation of integrals. As has been demonstrated, this made possible the solution of differential equations. Newton's second contribution was his formulation of the laws of mechanics. In particular, his second law, which says that force is equal to mass times acceleration, means that the study of motion can be reduced to a differential equation or to a system of differential equations. Finally, he discovered the universal law of gravity, which gave a mathematical description of the force of gravity. All of these results were published in 1687 in his *Philosophiae Naturalis Principia Mathematica* (*The Mathematical Principles of Natural Philosophy*), commonly referred to as the *Principia*.

Using his three discoveries, Newton was able to derive Kepler's laws of planetary motion. This means that for the first time there was a causal explanation of the motion of the planets. Newton's results were much broader in application, since they explained any kind of mechanical motion once the nature of the force was understood.

There were still difficulties with Newton's explanation. In particular, the force of gravity, as Newton described it, was a force acting at a distance. One body

[3] We have already discussed this briefly in Section 1.1 of Chapter 1.

acts on any other without any indication of a physical connection. Philosophers and physicists wondered how this was possible. In addition, by the end of the nineteenth century, some physical and mathematical anomalies had been observed. Although in most cases Newton's theory provided good answers, there were some situations in which the predictions of Newton's theory were not quite accurate.

These difficulties were apparently resolved in 1919, when Albert Einstein proposed his general theory of relativity. In this theory, gravity is explained as being the result of the curvature of four-dimensional space-time. This curvature in turn is caused by the masses of the bodies. The space-time itself provided the connection between the bodies and did away with problems of action at a distance. Finally, the general theory seems to have adequately explained most of the anomalies.

However, this is not the end of the story. Most physicists are convinced that all forces should be manifestations of one unified force. Early in the twentieth century they realized that there were four fundamental forces: gravity, the weak and strong nuclear forces, and electromagnetism. In the 1970s they were able to use quantum mechanics to unify the last three of these forces, but to date there is no generally accepted theory that unites gravity with the other three. There seems to be a fundamental conflict between general relativity and quantum mechanics.

A number of theories have been proposed to unify the two, but they remain unverified by experimental findings. Principal among these is *string theory*. The fundamental idea of string theory is that a particle is a tiny string that is moving in a 10-dimensional space-time. Four of these dimensions correspond to ordinary space-time. The extra six dimensions are assumed to have a tiny extent, on the order of 10^{-33} cm. This explains why these directions are not noticeable. It also gives a clue as to why string theory has no experimental verification. Nevertheless, as a theory it is very exciting. Hopefully someday it will be possible to devise an experimental test of the validity of string theory.

The modeling process

What we have described is a sequence of at least six different theories or mathematical models. The first were devised to explain the motion of the planets. Each was an improvement on the previous one, and starting with Newton they began to have more general application. With Newton's theory we have a model of all motion based on ordinary differential equations. His model was a complete departure from those that preceded it. It is his model that is used today, except when the relative velocities are so large that relativistic effects must be taken into account.

The continual improvement of the model in this case is what should take place wherever a mathematical model is used. As we learn more, we change the model to make it better. Furthermore, changes are always made on the basis of experimental findings that show faults in the existing model. The scientific theories of motion are probably the most mature of all scientific theories. Yet as our brief history shows, they are still being refined. This skepticism of the validity of existing theories is an important part of the scientific method. As good as our theories may seem, they can always be improved.

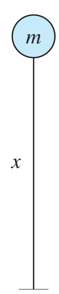


Figure 3. A ball near the surface of the earth.

Linear motion

Let's look now at Newton's theory of motion. We will limit ourselves for the moment to motion in one dimension. Think in terms of a ball that is moving only up and down near the surface of the earth, as shown in Figure 3. Recall that we have already discussed this in Sections 1.1 and 1.3 of Chapter 1.

To set the stage, we recall from Chapter 1 that the displacement x is the distance the ball is above the surface of the earth. Its derivative $v = x'$ is the velocity, and its second derivative $a = v' = x''$ is the acceleration. The mathematical model for motion is provided by Newton's second law. In our terms this is

$$F = ma, \tag{3.1}$$

where F is the force on the body and m is its mass. The gravitational force on a body moving near the surface of the earth is

$$F = -mg,$$

where g is the gravitational constant. It has value $g = 32$ ft/s^2 $= 9.8$ m/s^2. The minus sign is there because the direction of the force of gravity is always down, in the direction opposite to the positive x-direction. Thus, in this case, Newton's second law (3.1) becomes

$$m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = -mg, \quad \text{or} \quad \frac{dv}{dt} = \frac{d^2x}{dt^2} = -g. \tag{3.2}$$

We solved equation (3.2) in Example 3.14 in Section 1.3, and the solution is

$$x(t) = -\frac{1}{2}gt^2 + v_0t + x_0, \tag{3.3}$$

where v_0 is the original velocity and x_0 is the initial height.

Air resistance

In the derivation of our model in equation (3.2), we assumed that the only force acting was gravity. Now let's take into account the resistance of the air to the motion of the ball. If we think about how the resistance force acts, we come up with three simple facts. First, if there is no motion, then the velocity is zero, and there is no resistance. Second, the force always acts in the direction opposite to the motion. Thus if the ball is moving up, the resistance force is in the down direction, and if the ball is moving down, the force is in the up direction. From these considerations, we conclude that the resistance force has sign opposite to that of the velocity. We can put this mathematically by saying that the resistance force R has the form $R(x, v) = -r(x, v)v$, where r is a function that is always nonnegative. There are cases where r depends on x as well as v, such as when a ball is falling from a very high altitude so the density of the air has to be taken into account. However, in the cases we will consider r will depend only on v, so we will write

$$R(v) = -r(v)v. \tag{3.4}$$

Beyond these considerations, experiments have shown that the resistance force is somewhat complicated and there is no law that applies in all cases. Physicists use several models. We will look at two. In the first, resistance is proportional to the velocity, and in the second, the magnitude of the resistance is proportional to the square of the velocity. We will look at each of these cases in turn.

In the first case, r is a positive constant. Since forces add, our total force is the sum of the forces of gravity and air resistance,

$$F = -mg + R(v) = -mg - rv.$$

Using Newton's second law, we get

$$m\frac{dv}{dt} = -mg - rv, \quad \text{or} \quad \frac{dv}{dt} = -g - \frac{r}{m}v. \tag{3.5}$$

Notice that equation (3.5) is separable. Let's look for solutions. We separate variables to get

$$\frac{dv}{g + rv/m} = -dt.$$

When we integrate this and solve for v, we find the solution

$$v(t) = Ce^{-rt/m} - mg/r, \tag{3.6}$$

where C is a constant of integration.

We discover an interesting fact if we look at the limit of the velocity for large t. The exponential term in (3.6) decays to 0, so the velocity reaches the limit

$$\lim_{t\to\infty} v(t) = -\frac{mg}{r}.$$

Thus the velocity does not continue to increase in magnitude as the ball is falling. Instead it approaches the ***terminal velocity***

$$v_{\text{term}} = -mg/r. \tag{3.7}$$

We still have to solve for the displacement and for this we use equation (3.6), which we rewrite as

$$\frac{dx}{dt} = v = Ce^{-rt/m} - mg/r.$$

This equation can be solved by integration to get

$$x = -\frac{mC}{r}e^{-rt/m} - \frac{mgt}{r} + A,$$

where A is another constant of integration.

Example 3.8 Suppose you drop a brick from the top of a building that is 250 m high. The brick has a mass of 2 kg, and the resistance force is given by $R = -4v$. How long will it take the brick to reach the ground? What will be its velocity at that time?

The equation for the velocity of the brick is given in (3.6). Since we are dropping the brick, the initial condition is $v(0) = 0$, and we can use (3.6) to find that

$$0 = v(0) = C - mg/r \quad \text{or} \quad C = mg/r = 2 \times 9.8/4 = 4.9.$$

Then

$$\frac{dx}{dt} = v(t) = 4.9\left(e^{-2t} - 1\right). \tag{3.9}$$

Integrating, we get

$$x(t) = 4.9\left(-\frac{1}{2}e^{-2t} - t\right) + A.$$

The initial condition $x(0) = 250$ enables us to compute A, since evaluating the previous equation at $t = 0$ gives

$$250 = -\frac{4.9}{2} + A \quad \text{or} \quad A = 252.45.$$

Thus the equation for the height of the brick becomes

$$x(t) = 4.9\left(-\frac{1}{2}e^{-2t} - t\right) + 252.45.$$

We want to find t such that $x(t) = 0$. This equation cannot be solved using algebra, but a hand-held calculator or a computer can find a very accurate approximate solution. In this way we obtain $t = 51.5204$ seconds.

For a time this large the exponential term in (3.9) is negligible, so the brick has reached its terminal velocity of $v_{\text{term}} = -4.9$m/s. ●

Now let's turn to the second case, where the magnitude of the resistance force is proportional to the square of the velocity. Given the form of R in (3.4) together with the fact that $r \geq 0$, we see that the magnitude of R is

$$|R(v)| = r(v)|v| = kv^2$$

for some nonnegative constant k. Since $v^2 = |v|^2$, we conclude that $r = k|v|$, and the resistance force is $R(v) = -k|v|v$. In this case, Newton's second law becomes

$$m\frac{dv}{dt} = -mg - k|v|v, \quad \text{or} \quad \frac{dv}{dt} = -g - \frac{k}{m}|v|v. \tag{3.10}$$

Again, (3.10) is a separable equation. Let's look for solutions. Because of the absolute value, we have to consider separately the situation when the velocity is positive and the ball is moving upward and when the velocity is negative and the ball is descending. We will solve the equation for negative velocity and leave the other case to the exercises. When $v < 0$, $|v| = -v$, so (3.10) becomes

$$\frac{dv}{dt} = -g + \frac{k}{m}v^2. \tag{3.11}$$

Scaling variables to ease computation

We could solve (3.11) using separation of variables, but the constants cause things to get a little complicated. Instead, let's first introduce new variables by scaling the old ones. We introduce

$$v = \alpha w \quad \text{and} \quad t = \beta s,$$

where the constants α and β will be determined in a moment. Then

$$\frac{dv}{dt} = \frac{dv}{dw}\frac{dw}{ds}\frac{ds}{dt} = \frac{\alpha}{\beta}\frac{dw}{ds},$$

and equation (3.11) becomes

$$\frac{\alpha}{\beta}\frac{dw}{ds} = -g + \frac{k}{m}\alpha^2 w^2, \quad \text{or} \quad \frac{dw}{ds} = -\frac{g\beta}{\alpha} + \frac{k\alpha\beta}{m}w^2. \tag{3.12}$$

We choose α and β to make this equation as simple as possible. We require that

$$\frac{g\beta}{\alpha} = 1 \quad \text{and} \quad \frac{k\alpha\beta}{m} = 1.$$

Solving the first equation for α, we get $\alpha = g\beta$. Making this substitution in the second equation, it becomes $kg\beta^2/m = 1$. Solving for β and then for α, we get

$$\beta = \sqrt{\frac{m}{kg}} \quad \text{and} \quad \alpha = \sqrt{\frac{mg}{k}}.$$

As a reward for all of this, our differential equation in (3.12) simplifies to

$$\frac{dw}{ds} = -1 + w^2. \tag{3.13}$$

The separable equation (3.13) can be solved in the usual way. We first get

$$\frac{dw}{1 - w^2} = -ds.$$

Next we use partial fractions to write this as

$$\frac{1}{2}\left[\frac{dw}{1+w} + \frac{dw}{1-w}\right] = -ds.$$

This can be integrated to get

$$\frac{1}{2}\ln\left|\frac{1+w}{1-w}\right| = C - s,$$

where C is an arbitrary constant. When we exponentiate, we get

$$\left|\frac{1+w}{1-w}\right| = e^{2C-2s} = Ae^{-2s}.$$

By allowing A to be negative or 0, we see that in general

$$\frac{1+w}{1-w} = Ae^{-2s}.$$

Solving for w, we find that

$$w(t) = \frac{Ae^{-2s} - 1}{Ae^{-2s} + 1}.$$

In terms of our original variables v and t, this becomes

$$v(t) = -\sqrt{\frac{mg}{k}}\,\frac{1 - Ae^{-2t\sqrt{kg/m}}}{1 + Ae^{-2t\sqrt{kg/m}}}. \tag{3.14}$$

We want to observe the limiting behavior of $v(t)$ as $t \to \infty$. The exponential terms in (3.14) decay to 0, so the velocity approaches the terminal velocity

$$v_{\text{term}} = -\sqrt{mg/k}.$$

This should be compared to equation (3.7), which gives the terminal velocity when the air resistance is proportional to the velocity instead of to its square.

Finding the displacement

Integrating equation (3.14) to find the displacement is a daunting task to say the least. For certain problems the task can be made easier by eliminating the variable t from the equation $a = dv/dt$. This is done using the chain rule to write

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v. \tag{3.15}$$

Using this, equation (3.10) becomes

$$v\frac{dv}{dx} = -g - \frac{k}{m}|v|v. \tag{3.16}$$

Here is an example of how this can be useful.

Example 3.17 A ball of mass $m = 0.2$ kg is projected from the surface of the earth with velocity $v_0 = 50$ m/s. Assume that the force of air resistance is given by $R = -k\,|v|v$, where $k = 0.02$. What is the maximum height reached by the ball?

Since the ball is going up, the velocity is positive, so equation (3.16) becomes

$$v\frac{dv}{dx} = -g - \frac{k}{m}v^2 = -\frac{mg + kv^2}{m}.$$

When we separate variables, we get

$$\frac{v\,dv}{mg + kv^2} = -\frac{dx}{m}. \tag{3.18}$$

We will integrate this equation using the definite integral. To find what the end points of the integrations are, we notice first that at time $t = 0$ we have $x(0) = 0$, and $v(0) = v_0$. At a later time T, which is unknown and which need not be computed, the ball is at the top of its path, where $x(T) = x_{\max}$ and $v(T) = 0$. With these limits the integral of (3.18) is

$$\int_{v_0}^{0} \frac{v\,dv}{mg + kv^2} = -\int_0^{x_{\max}} \frac{dx}{m}.$$

Evaluating the integrals and solving for $x_{\max}$, we get

$$x_{\max} = \frac{m}{2k} \ln\left(1 + \frac{kv_0^2}{mg}\right).$$

With the data given we find that $x_{\max} = 16.4$ m. ●

EXERCISES

1. The acceleration due to gravity (near the earth's surface) is $9.8\,\text{m/s}^2$. If a rocketship in free space were able to maintain this constant acceleration indefinitely, how long would it take the ship to reach a speed equaling $(1/5)c$, where c is the speed of light? How far will the ship have traveled in this time? Ignore air resistance. *Note:* The speed of light is 3.0×10^8 m/s.
2. A balloon is ascending at a rate of 15 m/s at a height of 100 m above the ground when a package is dropped from the gondola. How long will it take the package to reach the ground? Ignore air resistance.
3. A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the

well? Ignore air resistance. *Note:* The speed of sound is 340 m/s.

4. A rocket is fired vertically and ascends with constant acceleration $a = 100\,\mathrm{m/s^2}$ for 1.0 min. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. Ignore air resistance.

5. A body is released from rest and travels the last half of the total distance fallen in precisely one second. How far did the body fall and how long did it take to fall the complete distance? Ignore air resistance.

6. A ball is projected vertically upward with initial velocity v_0 from ground level. Ignore air resistance.

(a) What is the maximum height acquired by the ball?

(b) How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?

(c) What is the speed of the ball when it impacts with the ground on its return?

7. A particle moves along a line with x, v, and a representing position, velocity, and acceleration, respectively. Assuming that the acceleration a is constant, use equation (3.15) to show that

$$v^2 = v_0^2 + 2a(x - x_0),$$

where x_0 and v_0 are the position and velocity of the particle at time $t = 0$, respectively. A car's speed is reduced from 60 mi/h to 30 mi/h in a span covering 500 ft. Calculate the magnitude and direction of the constant deceleration.

8. Near the surface of the earth, a ball is released from rest and its flight through the air offers resistance that is proportional to its velocity. How long will it take the ball to reach one-half of its terminal velocity? How far will it travel during this time?

9. A ball having mass $m = 0.1$ kg falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of 0.2 m/s, the force due to the resistance of the medium is -1 N. [One Newton [N] is the force required to accelerate a 1kg mass at a rate of $1\,\mathrm{m/s^2}$. Hence, $1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2}$.] Find the terminal velocity of the ball.

10. An object having mass 70 kg falls from rest under the influence of gravity. The terminal velocity of the object is -20 m/s. Assume that the air resistance is proportional to the velocity.

(a) Find the velocity and distance traveled at the end of 2 seconds.

(b) How long does it take the object to reach 80% of its terminal velocity?

11. A ball is thrown vertically into the air with unknown velocity v_0 at time $t = 0$. Assume that the ball is thrown from about shoulder height, say $y_0 = 1.5$ m. If you ignore air resistance, then it is easy to show that $dv/dt = -g$, where $g = 9.8\,\mathrm{m/s^2}$ is the acceleration due to gravity. Follow the lead of exercise 7 to show that $v\,dv = -g\,dy$. Further, because the velocity of the ball is zero when it reaches its maximum height,

$$\int_{v_0}^{0} v\,dv = \int_{1.5}^{15} -g\,dy.$$

Find the initial velocity of the ball if the ball reaches a maximum height of 15 m.

Next, let's include air resistance. Suppose that $R(v) = -rv$ and show that the equation of motion becomes

$$v\,dv = \left(-g - \frac{r}{m}v\right)\,dy.$$

If the mass of the ball is 0.1 kg and $r = 0.02\,\mathrm{N/(m/s)}$, find the initial velocity if the ball is again released from shoulder height ($y_0 = 1.5$ m) and reaches a maximum height of 15 m.

12. A mass of 0.2 kg is released from rest. As the object falls, air provides a resistance proportional to the velocity ($R(v) = -0.1v$), where the velocity is measured in m/s. If the mass is dropped from a height of 50 m, what is its velocity when it hits the ground? *Hint:* You may find equation (3.15) useful. Find v when $y = 0$.

13. An object having mass $m = 0.1$ kg is launched from ground level with an initial vertical velocity of 230 m/s. The air offers resistance proportional to the square of the object's velocity ($R(v) = -0.05v|v|$), where the velocity is measured in m/s. Find the maximum height acquired by the object.

14. One of the great discoveries in science is Newton's universal law of gravitation, which states that the magnitude of the gravitational force exerted by one point mass on another is proportional to their masses and inversely proportional to the square of the distance between them. In symbols,

$$|F| = \frac{GMm}{r^2},$$

where G is a universal gravitational constant. This constant, first measured by Lord Cavendish in 1798, has a currently accepted value approximately equal to $6.6726 \times 10^{-11}\,\mathrm{Nm^2/kg^2}$. Newton also showed that the law was valid for two spherical masses. In this case, you assume that the mass is concentrated at a point at the center of each sphere.

Suppose that an object with mass m is launched from the earth's surface with initial velocity v_0. Let y represent its position above the earth's surface, as shown in Figure 4.

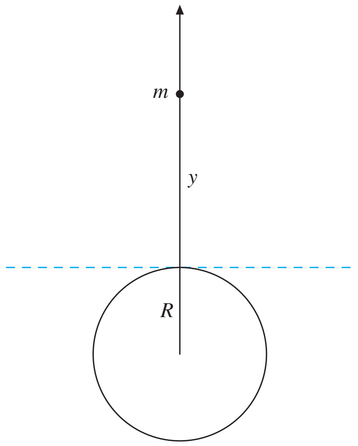


Figure 4. The object in Exercise 14.

(a) If air resistance is ignored, use the idea in equation (3.15) to help show that

$$v\frac{dv}{dy} = -\frac{GM}{(R+y)^2}. \tag{3.19}$$

(b) Assuming that $y(0) = 0$ (the object is launched from earth's surface) and $v(0) = v_0$, solve equation (3.19) to show that

$$v^2 = v_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right).$$

(c) Show that the maximum height reached by the object is given by

$$y = \frac{v_0^2 R}{2GM/R - v_0^2}.$$

(d) Show that the initial velocity

$$v_0 = \sqrt{\frac{2GM}{R}}$$

is the minimum required for the object to "escape" earth's gravitational field. *Hint:* If an object "escapes" earth's gravitational field, then the maximum height acquired by the object is potentially infinite.

15. Inside the earth, the surrounding mass exerts a gravitational pull in all directions. Of course, there is more mass towards the center of the earth than any other direction. The magnitude of this force is proportional to the distance from the center. (Can you prove this?) Suppose a hole is drilled to the center of the earth and a mass is dropped in the hole. Ignoring air resistance, with what velocity will the mass strike the center of the earth? As a hint, let x represent the distance of the mass from the center of the earth and note that equation (3.15) implies that the acceleration is $a = v(dv/dx)$.

16. An object with mass m is released from rest at a distance of a meters above the earth's surface (see Figure 5). Use Newton's universal law of gravitation (see Exercise 14) to show that the object impacts the earth surface with a velocity determined by

$$v = \sqrt{\frac{2agR}{a+R}},$$

where g is the acceleration due to gravity at the earth's surface and R is the radius of the earth. Ignore any affects due to the earth's rotation and atmosphere. *Hint:* On the earth's surface, explain why $mg = GMm/R^2$, where M is the mass of the earth and G is the universal gravitational constant.

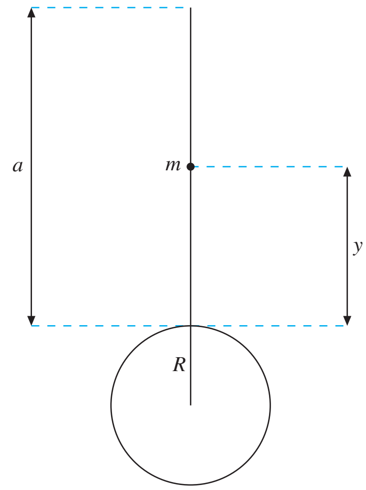


Figure 5. The object in Exercise 16.

17. A 2-foot length of a 10-foot chain hangs off the end of a high table. Neglecting friction, find the time required for the chain to slide off the table. *Hint*: Model this problem with a second-order differential equation and then solve it using the following reduction of order technique: If x is the length of the chain hanging off the table, then by equation (3.15) the acceleration is $a = v(dv/dx)$.

18. A parachutist of mass 60 kg free-falls from an airplane at an altitude of 5000 meters. He is subjected to an air resistance force that is proportional to his speed . Assume the constant of proportionality is 10 (kg/sec). Find and solve the differential equation governing the altitude of the parachuter at time t seconds after the start of his free-fall. Assuming he does not deploy his parachute, find his limiting velocity and how much time will elapse before he hits the ground.

19. In our models of air resistance the resistance force has depended only on the velocity. However, for an object that drops a considerable distance, such as the parachutist in the previous exercise, there is a dependence on the altitude as well. It is reasonable to assume that the resistance force is proportional to air pressure, as well as to the velocity. Furthermore, to a first approximation the air pressure varies exponentially with the altitude (i.e., it is proportional to e^{-ax}, where a is a constant and x is the altitude). Present a model using Newton's second law for the motion of an object in the earth's atmosphere subject to such a resistance force.