Math 3331 Differential Equations

1.1 Differential Equation Models

Blerina Xhabli

Department of Mathematics, University of Houston

blerina@math.uh.edu
math.uh.edu/∼blerina/teaching.html

Welcome

- Welcome to Math 3331 – Ordinary Differential Equations
- Course Info, Syllabus, Lecture Notes, and Homework are posted online:

  `math.uh.edu/~blerina/teaching.html`

What to Expect After the Course

You are expected to have three set of abilities and knowledge:

1. Pure Math: be able to solve equations and study the properties of solutions.
2. Applied Math: Know how to model a real world problem by ODEs
3. Computer: able to use MATLAB to study ODE numerically and visually.

1.1 Differential Equation Models

- Basic Idea of Using Differential Equations
- Motivating Examples
  - Newton's Second Law of mechanics
    - Newton's Universal Law of Gravitation
    - Newton's Model for the Motion of a Ball
    - Newton's Model of Planetary Motion
  - Population Growth Models
    - Exponential
    - Logistic
- Worked out Examples from Exercises:
  - 1.1, 1.2, 1.3, 1.4

Basic Idea of Using Differential Equations

Two Ways of Computing the Rate of Change

- In mathematics, the rate at which a quantity changes is the derivative of that quantity, e.g.

$$\frac{d\,y}{d\,t}, \quad \frac{d\,v}{d\,t} \quad \text{etc.}$$

- The second way of computing the rate of change comes from the application, and is different from one application to another.

When these two ways of computing the rate of change are equated, we get a differential equation.

Newton's Second Law of mechanics

Newton's Second Law of mechanics

The force acting on a mass is equal to the rate of change of momentum with respect to time.

$$\frac{d\,mv}{d\,t} = F.$$

Newton's Universal Law of Gravitation

Newton's Universal Law of Gravitation

Any body with mass M attracts any other body with mass m directly toward the mass M, with a magnitude proportional to the product of the two masses and inversely proportional to the square of the distance r separating them.

$$F = \frac{G\,M\,m}{r^2}, \quad G \text{ a universal constant.}$$

Newton's Model for the Motion of a Ball

Model for the Motion of a Ball Near the Surface of the Earth

Let x be the distance the ball is above the earth. Then

$$\frac{d^2 x}{d\,t^2} = -g,$$

where the constant g is the earth's gravitational acceleration

Newton's Model of Planetary Motion

Newton's Model of Planetary Motion

Let $\mathbf{x}(t)$ be the vector that gives the location of a planet relative to the sun. Then

$$m\frac{d^2\,\mathbf{x}}{d\,t^2} = -\frac{G\,M\,m}{|\mathbf{x}|^2}\,\frac{\mathbf{x}}{|\mathbf{x}|}.$$

Population Growth Models (I): Exponential

A Simple Model

The rate of change is proportional to the total population.

$$\frac{dP}{dt} = r\,P, \qquad r \text{ the reproductive rate constant.}$$

Show that the exponential function is a solution:

$$P(t) = P_0\, e^{rt}.$$

Population Growth Models (II): Logistic

A Better Model

A better model for the reproductive rate is $r(1 - P/K)$. Then

$$\frac{dP}{dt} = r\left(1 - P/K\right)P.$$

This ODE for $P(t)$, called the logistic equation, is much harder to solve, but it does a creditable job of predicting how single populations grow in isolated circumstances.

Basic Ideas for Exercises

Basic Idea

The phase "y is proportional to x" implies that y is related to x via the equation

$$y = kx,$$

where k is a constant.

In a similar manner,

- "y is proportional to the square of x" implies $y = kx^2$,
- "y is proportional to the product of x and z" implies $y = kxz$,
- "y is inversely proportional to the cube of x" implies $y = k/x^3$.

In all exercises, use these ideas to model each application with a differential equation. All rates are assumed to be with respect to time.

Exercise 1.1

Example (Exercise 1.1)

The rate of growth of bacteria in a petri dish is proportional to the number of bacteria in the dish.

$$y'(t) = k\,y(t).$$

Exercise 1.2

Example (Exercise 1.2)

The rate of growth of a population of eld mice is inversely proportional to the square root of the population.

$$y'(t) = k \,/\, \sqrt{y(t)}.$$

Exercise 1.3

Example (Exercise 1.3)

A certain area can sustain a maximum population of 100 ferrets. The rate of growth of a population of ferrets in this area is proportional to the product of the population and the difference between the actual population and the maximum sustainable population.

$$y'(t) = k\, y(t)\, (100 - y(t)).$$

Exercise 1.4

Example (Exercise 1.4)

The rate of decay of a given radioactive substance is proportional to the amount of substance remaining.

$$y'(t) = -k\, y(t).$$