Math 3331   Differential Equations

3.1 Modeling Population Growth

Blerina Xhabli

Department of Mathematics, University of Houston
blerina@math.uh.edu
math.uh.edu/~blerina/teaching.html
3.1 Modeling Population Growth

- Linear Model of Growth – Malthusian Model
  - Evaluating the Parameters
  - Models and the Real World

- Logistic Model of Growth
  - Solution of the Logistic Equation
  - Evaluating the Parameters in the Logistic Equation
  - Models and the Real World

- Worked out Examples from Exercises:
  - Linear Model of Growth: 2, 4
  - Logistic Model of Growth: 12, 14
Modeling Population Growth: Malthusian Model

- \( P(t) \): Population of species (bacteria, US-pop., ...)

- Model: \( \frac{dP}{dt} = f(P) \)

- Malthusian model:
  
  \[
  f(P) = r = b - d = \text{const}
  \]
  
  \( b \): birth rate, \( d \): death rate

  \[
  \Rightarrow \frac{dP}{dt} = rP
  \]

- Solution:
  
  \[
  P(t) = P_0 e^{rt}, \quad P_0 = P(0)
  \]

- \( r t = \ln[P(t)/P_0] \)

Use this to determine

- \( r \) if \( P_0, P(t_1) = P_1 \) are given

- \( t^* \) if \( r, P_0, P^* \) are given and \( t^* \) is sought s.t.
  
  \( P(t^*) = P^* \)
Example

**Ex.:** \( r = 0.1, \ P_0 = 10^3 \rightarrow \)

\[
P(10) = 2.7 \times 10^3 \\
P(100) = 22 \times 10^6
\]

(rapid growth \(\rightarrow\) see text p. 125)

**Figure 1** Exponential growth of a population.  
**Figure 2** Exponential growth over a longer period of time.
Example 3.1.4: Evaluating the Parameters

**Ex.:** At $t = 0$: $P_0 = 10$ cells. After 1 day: $P(1) = 25$ cells

Q: number of cells after 10 days?

$$r = \frac{1}{1} \ln(25/10) = 0.9163/\text{day} \; \Rightarrow \; P(10) = 10e^{10 \times 0.9613} \approx 95.4 \text{ cells}$$
Figure 1  Fitting a Malthusian model to early U.S. population.
Ex. 2: A cell culture is grown at \( t = 0 \).
After \( t_1 = 1 \) day: \( P_1 = 1000 \). After \( t_2 = 2 \) days: \( P_2 = 3000 \).
Q: \( P(0) = ? \)

\[
r(t_2 - t_1) = \ln(P_2/P_1) \Rightarrow r = (1/1) \ln(3000/1000) = 1.099/\text{day}
\]

\[
\Rightarrow P_0 = P(1)e^{-r \times 1} = 1000e^{-1.099} \approx 333
\]
Ex. 4: *Doubling Time:*

Given $t_d$ s.t. $P(t_d) = 2P_0 \Rightarrow P_0 e^{rt_d} = 2P_0 \Rightarrow rt_d = \ln 2 \Rightarrow r = (\ln 2)/t_d$

Q: Given $t_d = 10$ days and $P_0 = 1000$, find $t^*$ s.t. $P(t^*) = 10,000 \equiv P^*$

$t_d = 10$ days $\Rightarrow r = (\ln 2)/10 = 0.0693$/day

$\Rightarrow t^* = (1/r) \ln(P^*/P_0) = (\ln 10)/0.0693 \approx 33$ days
Modeling Population Growth: Logistic Model

Model: \( \frac{dP}{dt}/P = r - aP \)

- Set \( K = r/a \) \( \Rightarrow \)

\[
\frac{dP}{dt} = rP(1 - P/K) \equiv f(P)
\]

(1)

- Equilibria: \( P' = 0 \) \( \Rightarrow \)
  - \( P = 0 \): \( f'(0) = r > 0 \)
    \( \Rightarrow \) unstable
  - \( P = K \): \( f'(K) = -r < 0 \)
    \( \Rightarrow \) asympt. stable

Qualitative Analysis:

- \( K \): carrying capacity or eventual population
Solution of the Logistic Equation

Model: \( (dP/dt)/P = r - aP \)

- Set \( K = r/a \) \( \Rightarrow \)

\[
\frac{dP}{dt} = rP(1 - P/K) \equiv f(P) \tag{1}
\]

Solution of (1):

\[
P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}} \tag{2}
\]

Derivation of (2). S.o.V.: \( dP/[P(1 - P/K)] = [1/P - 1/(P - K)]dP = r \, dt \)

\[ \Rightarrow \ln |P| - \ln |K - P| = \ln |P/(K - P)| = rt + C \Rightarrow P/(K - P) = Ae^{rt} \]

For \( t = 0 : \) \( P_0/(K - P_0) = A \Rightarrow P_0(K - P)/[P(K - P_0)] = e^{-rt} \Rightarrow (2) \]
Evaluating the Parameters in the Logistic Equation

Computing Parameters:

- If \( K, P_0, t = h, P_1 = P(h) \) are known:

\[
P_1 = \frac{KP_0}{P_0 + (K - P_0)e^{-rh}}
\]

\[
\Rightarrow r = \frac{1}{h} \ln\left(\frac{P_1(K - P_0)}{P_0(K - P_1)}\right)
\]

- If \( P_0, t = h, P_1 = P(h), P_2 = P(2h) \) are known:

\[
r = \frac{1}{h} \ln\left(\frac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)}\right)
\]

\[
K = \frac{P_0P_1(1 - e^{-rh})}{P_0 - P_1e^{-rh}} = \frac{P_1P_2(1 - e^{-rh})}{P_1 - P_2e^{-rh}}
\]
Section 3.2: Models and the Real World

![Graph showing population growth over time](image)

**Figure 2** Fitting the logistic model to U.S. population.
Section 3.2: Models and the Real World

Figure 3 Logistic model projection of U.S. population.
Ex. 12: Given $K = 20,000$, $P_0 = 1000$ and $P_1 = P(8\text{ hrs}) = 1200$, find $r$, and $t^*$ s.t. $P(t^*) = \frac{3K}{4} = 15,000$.

$$r = \frac{1}{8} \ln\left(\frac{1.2(20 - 1)10^6}{1(20 - 1.2)10^6}\right)$$

$$\approx 0.0241/\text{hr}$$

$$P^* = \frac{KP_0}{[P_0 + (K - P_0)e^{-rt^*}]}$$

$$\Rightarrow t^* = \frac{1}{r} \ln\left(\frac{P^*(K - P_0)}{P_0(K - P^*)}\right)$$

$$= \frac{1}{0.0241} \ln\left(\frac{15(20 - 1)10^6}{1(20 - 15)10^6}\right)$$

$$\approx 72.22\text{ hrs}$$
Exercise 3.1.14

**Ex. 14 (modified):** Given $P_0 = 100$, $P_1 = P(20 \text{ hrs}) = 476$, $P_2 = P(40 \text{ hrs}) = 1986$, find $r$ and $K$.

\[
\begin{align*}
  r & = \frac{1}{20} \ln \left( \frac{1986(476 - 100)}{100(1986 - 476)} \right) \\
  & \approx 0.0799 \\
  \Rightarrow K & = \frac{476 \cdot 100(1 - e^{-0.08 \cdot 20})}{100 - 476e^{-0.08 \cdot 20}} \\
  & \approx 10,136
\end{align*}
\]