Math 3331 Differential Equations

3.1 Modeling Population Growth

Blerina Xhabli

Department of Mathematics, University of Houston

blerina@math.uh.edu
math.uh.edu/∼blerina/teaching.html

3.1 Modeling Population Growth

- Linear Model of Growth – Malthusian Model
  - Evaluating the Parameters
  - Models and the Real World
- Logistic Model of Growth
  - Solution of the Logistic Equation
  - Evaluating the Parameters in the Logistic Equation
  - Models and the Real World
- Worked out Examples from Exercises:
  - Linear Model of Growth: 2, 4
  - Logistic Model of Growth: 12, 14

Modeling Population Growth: Malthusian Model

- $P(t)$: Population of species (bacteria, US-pop., ...)
- Model: $\frac{dP/dt}{P} = f(P)$
- Malthusian model:

  $f(P) = r = b - d = \text{const}$

  b: birth rate, d: death rate

  $$\Rightarrow \quad \frac{dP}{dt} = rP$$
- Solution:

  $P(t) = P_0 e^{rt}, \; P_0 = P(0)$
- $\Rightarrow rt = \ln[P(t)/P_0]$

  Use this to determine
  - r if P_0, $P(t_1) = P_1$ are given
  - t^* if r, P_0, P^* are given and t^* is sought s.t. $P(t^*) = P^*$

Example

Ex.: $r = 0.1,\ P_0 = 10^3 \rightarrow$

$$\begin{aligned} P(10) &= 2.7 \times 10^3 \\ P(100) &= 22 \times 10^6 \end{aligned}$$

(rapid growth $\rightarrow$ see text p. 125)

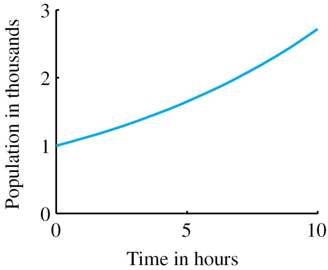


Figure 1 Exponential growth of a population.

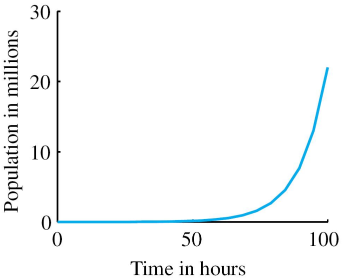


Figure 2 Exponential growth over a longer period of time.

Example 3.1.4: Evaluating the Parameters

Ex.: At $t = 0$: $P_0 = 10\,\text{cells}$. After 1 day: $P(1) = 25\,\text{cells}$
Q: number of cells after 10 days?

$$r = (1/1)\ln(25/10) = 0.9163/\text{day} \;\Rightarrow\; P(10) = 10e^{10\times 0.9613} \approx 95.4\,\text{cells}$$

Section 3.2: Models and the Real World

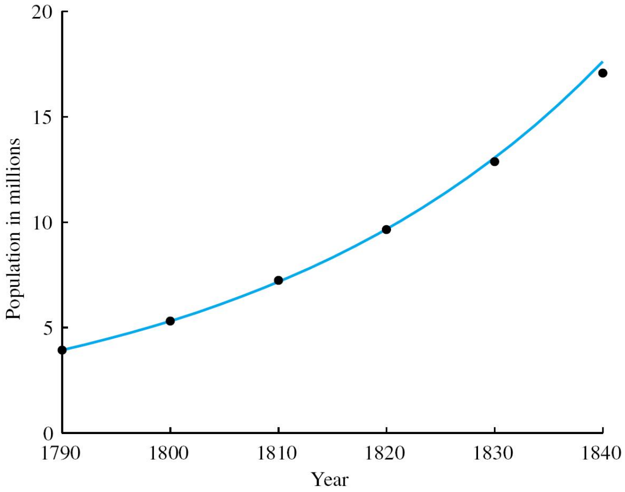


Figure 1 Fitting a Malthusian model to early U.S. population.

Exercise 3.1.2

Ex. 2: A cell culture is grown at $t = 0$.
After $t_1 = 1$ day: $P_1 = 1000$. After $t_2 = 2$ days: $P_2 = 3000$.
Q: $P(0) = ?$

$$r(t_2 - t_1) = \ln(P_2/P_1) \Rightarrow r = (1/1)\ln(3000/1000) = 1.099/\text{day}$$

$$\Rightarrow P_0 = P(1)e^{-r\times 1} = 1000e^{-1.099} \approx 333$$

Exercise 3.1.4

Ex. 4: *Doubling Time:*

Given t_d s.t. $P(t_d) = 2P_0 \Rightarrow P_0 e^{rt_d} = 2P_0 \Rightarrow rt_d = \ln 2 \Rightarrow r = (\ln 2)/t_d$

Q: Given $t_d = 10$ days and $P_0 = 1000$, find t^* s.t. $P(t^*) = 10,000 \equiv P^*$

$$t_d = 10 \text{ days} \Rightarrow r = (\ln 2)/10 = 0.0693/\text{day}$$

$$\Rightarrow t^* = (1/r)\ln(P^*/P_0) = (\ln 10)/0.0693 \approx 33 \text{ days}$$

Modeling Population Growth: Logistic Model

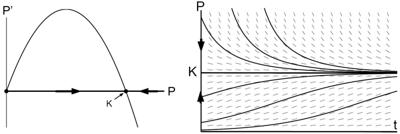
Model: $(dP/dt)/P = r - aP$

- Set $K = r/a \Rightarrow$

$$\frac{dP}{dt} = rP(1 - P/K) \equiv f(P) \qquad (1)$$

- Equilibria: $P' = 0 \Rightarrow$
  $P = 0$: $f'(0) = r > 0$
  $\Rightarrow$ unstable
  $P = K$: $f'(K) = -r < 0$
  $\Rightarrow$ asympt. stable

Qualitative Analysis:



K: carrying capacity or eventual population

Solution of the Logistic Equation

Model: $(dP/dt)/P = r - aP$

- Set $K = r/a \Rightarrow$

$$\frac{dP}{dt} = rP(1 - P/K) \equiv f(P) \qquad (1)$$

Solution of (1):

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}} \qquad (2)$$

Derivation of (2). S.o.V.: $dP/[P(1-P/K)] = [1/P - 1/(P-K)]dP = r\,dt$
$\Rightarrow \ln|P| - \ln|K-P| = \ln|P/(K-P)| = rt + C \Rightarrow P/(K-P) = Ae^{rt}$
For $t = 0:\ P_0/(K-P_0) = A \Rightarrow P_0(K-P)/[P(K-P_0)] = e^{-rt} \Rightarrow$ (2)

Evaluating the Parameters in the Logistic Equation

Computing Parameters:

- If K, P_0, $t = h$, $P_1 = P(h)$ are known:

$$P_1 = \frac{KP_0}{P_0 + (K - P_0)e^{-rh}}$$

$$\Rightarrow r = \frac{1}{h}\ln\left(\frac{P_1(K - P_0)}{P_0(K - P_1)}\right)$$

- If P_0, $t = h$, $P_1 = P(h)$, $P_2 = P(2h)$ are known:

$$r = \frac{1}{h}\ln\left(\frac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)}\right)$$

$$K = \frac{P_0P_1(1 - e^{-rh})}{P_0 - P_1e^{-rh}} = \frac{P_1P_2(1 - e^{-rh})}{P_1 - P_2e^{-rh}}$$

Section 3.2: Models and the Real World

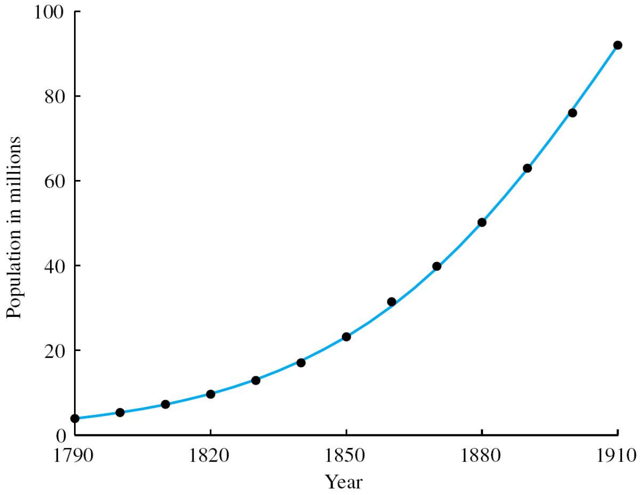


Figure 2 Fitting the logistic model to U.S. population.

Section 3.2: Models and the Real World

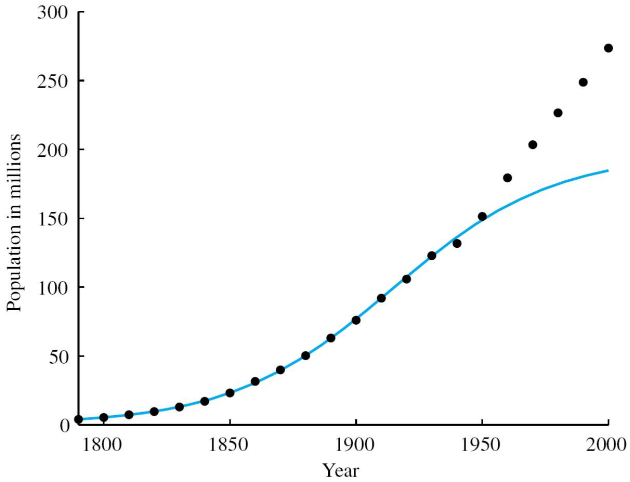


Figure 3 Logistic model projection of U.S. population.

Exercise 3.1.12

Ex. 12: Given $K = 20,000$, $P_0 = 1000$ and $P_1 = P(8\,\text{hrs}) = 1200$, find r, and t^* s.t. $P(t^*) = 3K/4 = 15,000$.

$$\begin{aligned} r &= (1/8)\ln\left(\frac{1.2(20-1)10^6}{1(20-1.2)10^6}\right) \\ &\approx 0.0241/\text{hr} \end{aligned}$$

$$\begin{aligned} P^* &= KP_0/[P_0 + (K-P_0)e^{-rt^*}] \\ \Rightarrow t^* &= (1/r)\ln\left(\frac{P^*(K-P_0)}{P_0(K-P^*)}\right) \\ &= \frac{1}{0.0241}\ln\left(\frac{15(20-1)10^6}{1(20-15)10^6]}\right) \\ &\approx 72.22\,\text{hrs} \end{aligned}$$

Exercise 3.1.14

Ex. 14 (modified): Given $P_0 = 100$, $P_1 = P(20\,\text{hrs}) = 476$, $P_2 = P(40\,\text{hrs}) = 1986$, find r and K.

$$\begin{aligned} r &= \frac{1}{20}\ln\Big(\frac{1986(476-100)}{100(1986-476)}\Big) \\ &\approx 0.0799 \\ \Rightarrow K &= \frac{476\cdot 100(1-e^{-0.08\cdot 20})}{100-476e^{-0.08\cdot 20}} \\ &\approx 10,136 \end{aligned}$$