Math 3331 Differential Equations

4.1 Second-Order Equations

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4.1 Second-Order Equations

- Second-Order Equation: Models
  - Vibrating Spring
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- General Solution
  - Solution Structure
  - Linear Independence and Wronskian
  - Existence and Uniqueness
- Worked out Examples from Exercises
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Definition

Second-Order Equation

$$y'' = f(t, y, y')$$

Linear Equation

$$y'' + p(t)y' + q(t)y = g(t)$$

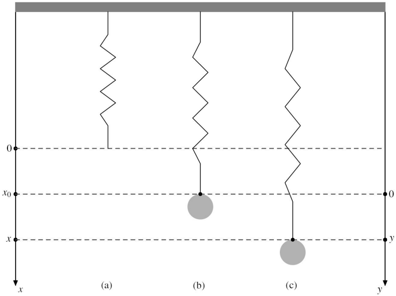
where the coefficients $p(t)$, $q(t)$ and $g(t)$ are functions of t.

Homogeneous Equation

$$y'' + p(t)y' + q(t)y = 0$$

that is, the forcing term $g(t)$ is equal to 0.

Example: Vibrating Spring



- Hooke's low: $R(x) = -kx$ with k the spring constant.
- Spring-mass equilibrium: $R(x_0) + mg = 0$.

Newton's second law:

$$mx'' = mg + R(x) + D(x') + F(t)$$

where

- mg is the force of gravity,
- $R(x)$ the restoring force of the spring,
- $D(x')$ a damping force, and
- $F(t)$ is an external force.

Let $y = x - x_0$ the displacement.

$$my'' = -ky + D(y') + F(t)$$

Example: Vibrating Spring with Damping

Let the damping force
$D(y') = -\mu y'$
with μ the dampling constant.

The 2nd order linear DE for y

$$my'' + \mu y' + ky = F(t)$$

For undamped $\mu = 0$ and unforced $F(t) = 0$ spring, the DE reduces to the harmonic equation

$$y'' + \omega_0^2 y = 0$$

with $\omega_0 = \sqrt{k/m}$ the natural frequency.

The general solution to the harmonic equation is

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

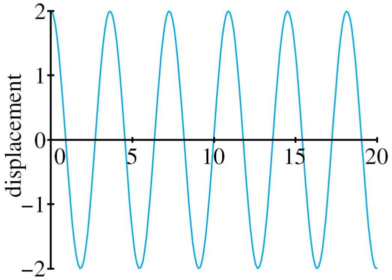


Figure 2 A vibrating spring with no damping.

Structure of the General Solution

Theorem 1.23

Suppose that y_1 and y_2 are linearly independent solutions to the equation

$$y'' + p(t)y' + q(t)y = 0.$$

Its general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

where C_1 and C_2 are arbitrary constants.

It can be shown that

$$y_1(t) = \cos(\omega_0 t) \text{ and } y_2(t) = \sin(\omega_0 t)$$

are linearly independent solutions to the harmonic equation

$$y'' + \omega_0^2 y = 0$$

Linear Independence and Wronskian

Definition 1.22

Two functions u and v are linearly independent on the interval (α, β) if neither is a constant multiple of the other on that interval.

Proposition 1.27

Suppose that u and v are solutions to the equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β). Then u and v are linearly independent if and only if their Wronskian

$$W(t) = \det \begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix} = u(t)v'(t) - v(t)u'(t)$$

never vanishes in (α, β), i.e., $W(t_0) \neq 0$ for some t_0 in (α, β).

IVP and EUT

Theorem 1.17 (Existence and Uniqueness of Solution)

Suppose that $p(t)$,$q(t)$, and $g(t)$ are continuous on (α, β). Let $t_0 \in (\alpha, \beta)$. Then for any real numbers y_0 and y_1, there is one and only one function $y(t)$ defined on (α, β), which is a solution to the the initial value problem

$$y'' + p(t)y' + q(t)y = g(t) \quad \text{for } \alpha < t < \beta$$

with the initial conditions

$$y(t_0) = y_0, \quad \text{and} \quad y'(t_0) = y_1.$$

Example 1.31

Example

Find the solution to the harmonic equation $x'' + 4x = 0$ with intial conditions $x(0) = 4$ and $x'(0) = 2$.

We know from Example 1.24 that the general solution has the form

$$x(t) = a\cos 2t + b\sin 2t,$$

where a and b are arbitrary constants. Substituting the initial conditions we get

$$4 = x(0) = a, \quad \text{and} \quad 2 = x'(0) = 2b.$$

Thus $a = 4$ and $b = 1$ and our solution is

$$x(t) = 4\cos 2t + \sin 2t.$$

Exercise 4.1.2

Determine whether the equation

$$t^2 y'' = 4y' - \sin t$$

is linear or nonlinear. If linear, state whether it is homogeneous or inhomogeneous.

Divide both sides of $t^2 y'' = 4y' - \sin t$ by t^2, then rearrange to obtain

$$y'' - \frac{4}{t^2} y' = -\frac{\sin t}{t^2}.$$

Compare this with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = -4/t^2$, $q(t) = 0$, and $g(t) = -(\sin t)/t^2$. Hence, the equation is linear and inhomogeneous.

Exercise 4.1.4

Determine whether the equation

$$ty'' + (\sin t)\, y' = 4y - \cos 5t$$

is linear or nonlinear. If linear, state whether it is homogeneous or inhomogeneous.

Divide both sides of $ty'' + (\sin t)y' = 4y - \cos 5t$ by t, then rearrange to obtain

$$y'' + \frac{\sin t}{t} y' - \frac{4}{t} = -\frac{\cos 5t}{t}$$

Compare this with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = (\sin t)/t$, $q(t) = -4/t$, and $g(t) = -(\cos 5t)/t$. Hence, the equation is linear and inhomogeneous.

Exercise 4.1.22

Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for

$$y'' + 2y' - 3y = 0,$$

then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.

If $y_1(t) = e^t$, then

$$y'' + 2y' - 3y = e^t + 2e^t - 3e^t = 0,$$

and if $y_2(t) = e^{-3t}$, then

$$y'' + 2y' - 3y = 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0,$$

Furthermore,

$$\frac{y_1(t)}{y_2(t)} = \frac{e^t}{e^{-3t}} = e^{4t},$$

which is nonconstant. Thus, y_1 is not a constant multiple of y_2 and the solutions $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions.

Thus, the general solution of $y'' + 2y' - 3y = 0$ is

$$y(t) = C_1 e^t + C_2 e^{-3t},$$

and its derivative is

$$y'(t) = C_1 e^t - 3C_2 e^{-3t} t.$$

The initial conditions, $y(0) = 1$ and $y'(0) = -2$ lead to the equations

$$\begin{aligned} 1 &= C_1 + C_2 \\ -2 &= C_1 - 3C_2 \end{aligned}$$

and the constants $C_1 = 1/4$ and $C_2 = 3/4$. Thus, the solution of the initial value problem is

$$y(t) = \frac{1}{4}e^t + \frac{3}{4}e^{-3t}.$$

Exercise 4.1.24

Show that $y_1(t) = e^{-t}\cos 2t$ and $y_2(t) = e^{-t}\sin 2t$ form a fundamental set of solutions for

$$y'' + 2y' + 5y = 0,$$

then find a solution satisfying $y(0) = -1$ and $y'(0) = 0$.

If $y_1(t) = e^{-t}\cos 2t$, then

$$y_1'(t) = -e^{-t}\cos 2t - 2e^{-t}\sin 2t, \quad \text{and}$$
$$y_1''(t) = -3e^{-t}\cos 2t + 4e^{-t}\sin 2t.$$

Thus,

$$\begin{aligned} y_1'' + 2y_1' + 5y_1 &= -3e^{-t}\cos 2t + 4e^{-t}\sin 2t \\ &\quad - 2e^{-t}\cos 2t - 4e^{-t}\sin 2t + 5e^{-t}\cos 2t \\ &= 0. \end{aligned}$$

If $y_2(t) = e^{-t}\sin 2t$, then

$$y_2'(t) = -e^{-t}\sin 2t + 2e^{-t}\cos 2t, \quad \text{and}$$
$$y_2''(t) = -3e^{-t}\sin 2t - 4e^{-t}\cos 2t.$$

Thus,

$$\begin{aligned} y_2'' + 2y_2' + 5y_2 &= -3e^{-t}\sin 2t - 4e^{-t}\cos 2t \\ &\quad - 2e^{-t}\sin 2t + 4e^{-t}\cos 2t + 5e^{-t}\sin 2t \\ &= 0. \end{aligned}$$

Furthermore,

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-t}\cos 2t}{e^{-t}\sin 2t} = \cot 2t,$$

which is nonconstant. Thus, y_1 is not a constant multiple of y_2 and the solutions $y_1(t) = e^{-t}\cos 2t$ and $y_2(t) = e^{-t}\sin 2t$ form a fundamental set of solutions. Thus, the general solution of $y''+2y'+5y = 0$ is

$$y(t) = C_1e^{-t}\cos 2t + C_2e^{-t}\sin 2t,$$

and its derivative is

$$\begin{aligned} y'(t) &= -C_1e^{-t}\cos 2t - 2C_1e^{-t}\sin 2t \\ &\quad - C_2e^{-t}\sin 2t + 2C_2e^{-t}\cos 2t. \end{aligned}$$

The initial conditions, $y(0) = -1$ and $y'(0) = 0$ lead to the equations

$$\begin{aligned} -1 &= C_1 \\ 0 &= -C_1 + 2C_2 \end{aligned}$$

and the constants $C_1 = -1$ and $C_2 = -1/2$. Thus, the solution of the initial value problem is

$$y(t) = -e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t.$$