Math 3331  Differential Equations

4.2 Second-Order Equations and Systems

Blerina Xhabli

Department of Mathematics, University of Houston

blerina@math.uh.edu
math.uh.edu/~blerina/teaching.html
4.2 Second-Order Equations and Systems

- Second-Order Equations
- Planar Systems
  - yv-Phase Plane Plot
  - Phase Plane Portrait
Second-Order Equations and Planar Systems

Second-order DE

\[ y'' + ay' + by = 0 \]  
\[ p(\lambda) = \lambda^2 + a\lambda + b = 0 \]  

planar system

\[ x_1 = y, \quad x_2 = v = y' \]  
\[ x' = Ax, \quad A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \]  
\[ \text{det}(A - \lambda I) = p(\lambda) \]  

\[(\text{Chapter 9})\]

\[ y'v\text{-Phase Plane Plot} \]

A damped unforced spring:

\[ my'' + \mu y' + ky = 0 \]

with \( m = 1, \mu = 0.4, \) and \( k = 3. \)
**Phase Plane Portrait**

**Ex.:** \( y'' - y = 0 \) \( (a = 0, \ b = -1) \)

\[ p(\lambda) = \lambda^2 - 1 \Rightarrow \lambda = \pm 1 \text{ (saddle)} \]

General solution: \( y(t) = c_1 e^t + c_2 e^{-t} \)

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{cases} 
\lambda_1 = 1 & \iff v_1 = [1, 1]^T \\
\lambda_2 = -1 & \iff v_2 = [-1, 1]^T 
\end{cases}
\]

Phase plane portrait for DE (1) = Phase plane portrait for (2)
**Ex.:** \( y'' - 3y' + 2y = 0 \)

\[ p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) \]

\[ \Rightarrow \text{source: } y(t) = c_1 e^t + c_2 e^{2t} \]

\[ A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \rightarrow \begin{cases} \lambda_1 = 1 \leftrightarrow v_1 = [1, 1]^T \\ \lambda_2 = 2 \leftrightarrow v_2 = [1, 2]^T \end{cases} \]