Math 3331 Differential Equations

4.4 Harmonic Motion

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4.4 Harmonic Motion

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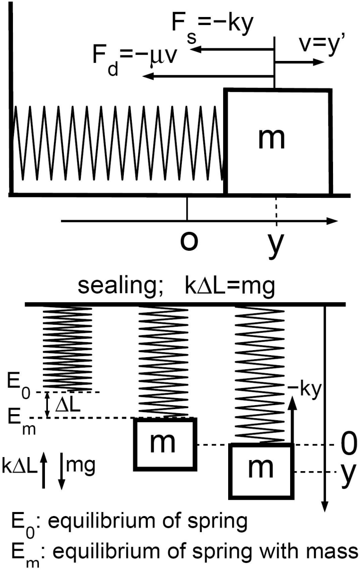
Harmonic Motion: Mass-Spring System

Mass–spring system:

$$my'' + \mu y' + ky = 0$$

- m: mass (kg)
- μ: damping constant (kg/s)
- k: spring constant (kg/s^2)
- y: deviation of mass position from equilibrium position (m)
- y': velocity (m/s)



Harmonic Motion: Pendulum For Small ϕ

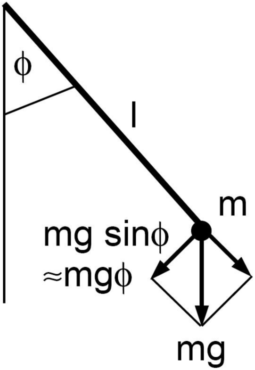
Pendulum for small ϕ:

$$\phi'' + (\mu/m)\phi' + (g/l)\phi = 0$$

ϕ: angle (no unit)

g: $9.8\, m/s^2$

l: length (m)

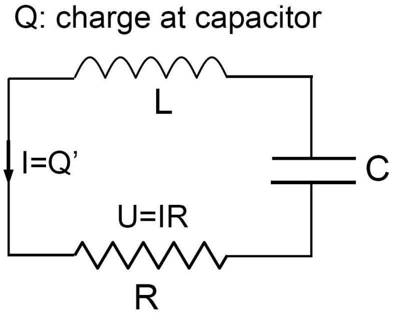


Harmonic Motion: RLC-Circuit

RLC–circuit:

$$LQ'' + RQ' + Q/C = 0$$

- Q: charge (C)
- I: current Q' (A)
- L: inductivity (H)
- R: resistor (Ω)
- C: capacity (F)



Classification of Harmonic Motion: Mass-Spring System

Mass-Spring System

DE:

$$y'' + \frac{\mu}{m}y' + \frac{k}{m}y = 0,$$

$$m > 0, \quad k > 0, \quad \mu \geq 0$$

Characteristic Eqn:

$$\lambda^2 + \frac{\mu}{m}\lambda + \frac{k}{m} = 0$$

Roots:

$$\lambda_{1,2} = -\frac{\mu}{2m} \pm \frac{1}{2m}\sqrt{\mu^2 - 4km}$$

Classification

1. Undamped Case:

$$\mu = 0$$

2. Underdamped Case:

$$0 < \mu^2 < 4km$$

3. Critically damped Case:

$$\mu^2 = 4km$$

4. Overdamped Case:

$$\mu^2 > 4km$$

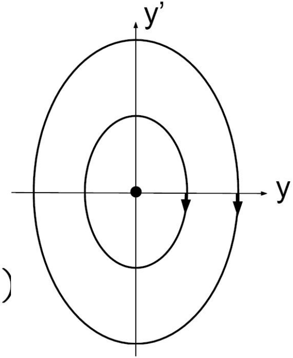
Mass-Spring System: Undamped Case ($\mu = 0$)

Undamped Case: $\mu = 0$

$$\lambda = \pm i\omega_0$$

$$\omega_0 = \sqrt{k/m}$$

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$



- oscillation
- phase portrait: center
- clockwise direction of rotation

Mass-Spring System: Underdamped Case ($0 < \mu^2 < 4km$)

Underdamped Case: $0 < \mu^2 < 4km$

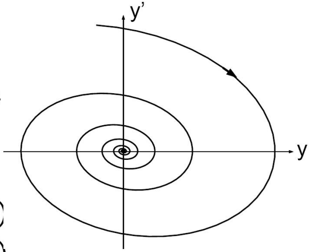
$$\lambda_{1,2} = -\alpha \pm i\omega$$

$$\alpha = \mu/2m$$

$$\omega = \sqrt{4km - \mu^2}/(2m$$

$$= \sqrt{\omega_0^2 - \mu^2/4m^2}$$

$$y(t) = e^{-\alpha t}(c_1 \cos(\omega t) + c_2 \sin(\omega t))$$



- damped oscillation
- phase portrait: spiral sink
- clockwise direction of rotation

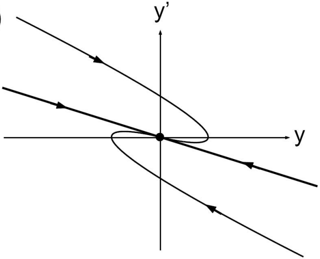
Mass-Spring System: Critically damped Case ($\mu^2 = 4km$)

Critically Damped Case: $\mu^2 = 4km$

$\lambda_1 = \lambda_2 = -\mu/(2m)$

$y(t) = e^{\lambda_1 t}(c_1 + c_2 t)$

- phase portrait: degenerate nodal sink



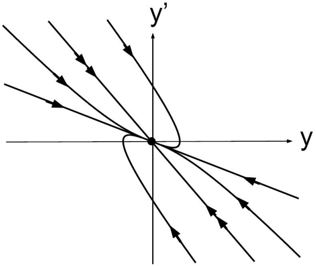
Mass-Spring System: Overdamped Case ($\mu^2 > 4km$)

Overdamped Case: $\mu^2 > 4km$

$$\lambda_1 < \lambda_2 < 0$$

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

- phase portrait: nodal sink
- both eigenlines: negative slopes



Harmonic Motion: Undamped Case

A: **amplitude**

ϕ: **phase angle**, choose $-\pi < \phi \leq \pi$

$$\phi = \begin{cases} \arctan(c_2/c_1) & \text{if } c_1 > 0 \\ \arctan(c_2/c_1) + \pi & \text{if } c_1 < 0,\ c_2 \geq 0 \\ \arctan(c_2/c_1) - \pi & \text{if } c_1 < 0,\ c_2 < 0 \\ \pi/2 & \text{if } c_1 = 0,\ c_2 > 0 \\ -\pi/2 & \text{if } c_1 = 0,\ c_2 < 0 \end{cases}$$

Undamped Case:

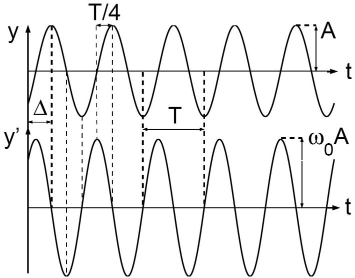
$$\begin{aligned} y(t) &= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \\ &= A[d_1 \cos(\omega_0 t) + d_2 \sin(\omega_0 t)] \end{aligned}$$

where $\begin{cases} A = \sqrt{c_1^2 + c_2^2} \\ d_1 = c_1/A,\ d_2 = c_2/A \end{cases}$

Since $d_1^2 + d_2^2 = 1$ we can define ϕ by

$$d_1 = \cos\phi, \quad d_2 = \sin\phi$$
$$\Rightarrow \quad d_2/d_1 = c_2/c_1 = \tan\phi$$

$$\begin{aligned} y(t) &= A[\cos\phi \cos(\omega_0 t) + \sin\phi \sin(\omega_0 t)] \\ &= A\cos(\omega_0 t - \phi) \\ y'(t) &= -\omega_0 A \sin(\omega_0 t - \phi) \end{aligned}$$



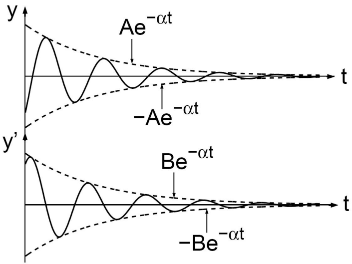
$\Delta = (\phi/2\pi)T, \quad T = 2\pi/\omega_0$

Harmonic Motion: Underdamped Case

Underdamped Case:

$$\begin{aligned} y(t) &= e^{-\alpha t}[c_1 \cos(\omega t) + c_2 \sin(\omega t)] \\ &= e^{-\alpha t} A \cos(\omega t - \phi) \\ y'(t) &= e^{-\alpha t}[(\omega c_2 - \alpha c_1)\cos(\omega t) \\ &\qquad -(\omega c_1 + \alpha c_2)\sin(\omega t)] \\ &= e^{-\alpha t} B \cos(\omega t - \psi) \end{aligned}$$

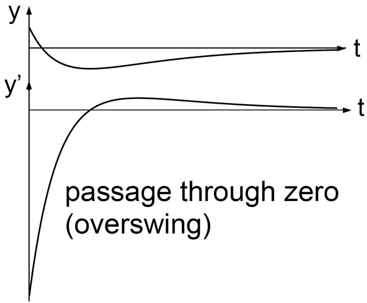
$\pm Ae^{-\alpha t}$, $\pm Be^{-\alpha t}$: envelopes of damped oscillations

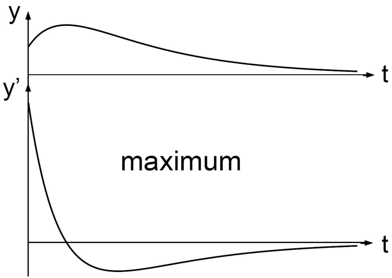


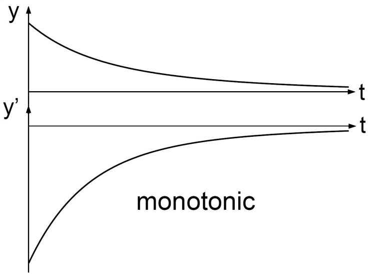
Harmonic Motion: Critically and Overdamped Cases

Critically and Overdamped Cases:

- If $y(0)$ and $y'(0)$ have equal signs, then
  - $y(t)$ attains maximum or minimum
  - $y'(t)$ crosses zero
- If $y(0)$ and $y'(0)$ have opposite signs, then $y(t)$
  - crosses zero if $|y'(0)/y(0)|$ is large
  - is monotonic if $|y'(0)/y(0)|$ is small







Exercise 4.4.11

Ex. 4.4.11: Given an undamped mass-spring system with $m = 0.2\,kg$, $k = 5\,kg/s^2$, $y(0) = 0.5\,m$, $y'(0) = 0$, find amplitude, frequency, phase of motion.

Natural frequency: $\omega_0 = \sqrt{5/0.2} = 5/s \Rightarrow$

$$y(t) = c_1 \cos 5t + c_2 \sin 5t, \quad y'(t) = -5c_1 \sin 5t + 5c_2 \cos 5t$$
$$\text{IC: } y(0) = c_1 = 0.5, \; y'(0) = 5c_2 = 0 \;\Rightarrow\; y(t) = 0.5 \cos 5t$$
$$\Rightarrow \text{ amplitude: } A = 0.5\,m, \quad \text{phase: } \phi = 0$$

Exercise 4.4.22

Ex. 4.4.22: A mass-spring system with $m = 0.1\,kg$, $k = 9.8\,kg/s^2$ is placed in a viscous medium with friction force $0.3\,N$ if $v = 0.2\,m/s$. Initial data: $y(0) = 0.1\,m$, $y'(0) = 0$. Find amplitude, frequency, and phase of motion.

Friction coefficient: $F_d = \mu v \Rightarrow 0.3 = \mu 0.2 \Rightarrow \mu = 1.5\,kg/s$.

$$\text{ODE:}\quad 0.1y'' + 1.5y' + 9.8y = 0 \Rightarrow y'' + 15y' + 98y = 0 \Rightarrow$$

$p(\lambda) = \lambda^2 + 15\lambda + 98 = (\lambda+7.5)^2 + 41.75 \Rightarrow \lambda = -7.5 \pm i\sqrt{41.75} \approx -7.5 \pm 6.461i$

$\Rightarrow$ Damped motion with frequency $\omega \approx 6.461\,/s$ of harmonic part $\Rightarrow$

$$y(t) = e^{-7.5t}(c_1 \cos\omega t + c_2 \sin\omega t)$$

$$y'(t) = e^{-7.5t}[(\omega c_2 - 7.5c_1)\cos\omega t - (\omega c_1 + 7.5c_2)\sin\omega t]$$

$$\text{IC:}\quad y(0) = c_1 = 0.1,\quad y'(0) = \omega c_2 - 7.5c_1 = 0 \quad\Rightarrow\quad c_2 = 7.5c_1/\omega \approx 0.116$$

$$\Rightarrow\ y(t) = e^{-7.5t}(0.1\cos\omega t + 0.116\sin\omega t)$$

Exercise 4.4.22 (cont.)

Amplitude of harmonic part: $A_0 \approx \sqrt{0.1^2 + 0.116^2} \approx 0.153$. Since $c_1, c_2 > 0$
$\Rightarrow$ phase angle $\phi = \arctan(c_2/c_1) \approx \arctan(1.16) \approx 0.859$

$$\Rightarrow y(t) = 0.153e^{-7.5t}\cos(6.461t - 0.859)$$

$\Rightarrow$ amplitude: $A(t) = 0.153e^{-7.5t}\,m$,
frequency: $\omega = 6.461/s$, phase: $\phi = 0.859$