Math 3331 Differential Equations

4.6 Variation of Parameters

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- Second-Order Equations and Planar Systems
- Fundamental Set of Solutions and Fundamental Matrix
- Variation of Parameters for Systems
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Second-Order Equations and Planar Systems

Inhomogeneous equation:

$$y'' + a(t)y' + b(t)y = F(t) \quad (8)$$

Homogeneous equation:

$$y'' + a(t)y' + b(t)y = 0 \quad (9)$$

Inhomogeneous system:

$$\mathbf{x}' = A(t)\mathbf{x} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \quad (10)$$

where $A(t) = \begin{bmatrix} 0 & 1 \\ -b(t) & -a(t) \end{bmatrix}$

Fundamental Set of Solutions and Fundamental Matrix

Let $y_1(t), y_2(t)$ be F.S.S for (9)

$$\Rightarrow X(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$$

is F.M. of $\mathbf{x}' = A(t)\mathbf{x}$, and

$$X(t)^{-1} = \frac{1}{W(t)} \begin{bmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{bmatrix}$$

where $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$.

A matrix-valued function $X(t)$ is a fundamental matrix for the system $x' = A(t)x$ if and only if

$$X' = A(t)X, \quad X_0 = X(t_0) \text{ is invertible for some } t_0$$

Variation of Parameters for Systems

Look for a solution of the form

$$x_p(t) = X(t)v(t), \quad v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$$

Differentiating it gives

$$x_p' = X'v + Xv' = A(t)Xv + Xv' = A(t)x_p + Xv'$$

x_p is to be a solution to $x_p' = A(t)x_p + f(t)$ provided that

$$Xv' = f(t) \quad \Rightarrow \quad v' = X^{-1}f(t) \quad \Rightarrow \quad v(t) = \int X^{-1}(t)f(t)dt$$

Inserting v gives the particular solution

$$x_p(t) = X(t)\left(\int X^{-1}(t)f(t)dt\right)$$

Variation of Parameters for Systems (cont.)

$$\text{Set} \quad \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \int X(t)^{-1} \begin{bmatrix} 0 \\ F(t) \end{bmatrix} dt$$

$$\Rightarrow \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \int \begin{bmatrix} -y_2(t) \\ y_1(t) \end{bmatrix} \frac{F(t)}{W(t)} dt$$

$$v_1(t) = -\int y_2(t) \frac{F(t)}{W(t)} dt$$

$$v_2(t) = \int y_1(t) \frac{F(t)}{W(t)} dt$$

(recall that $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$)

Particular Solution to System

Particular solution of (10):

$$\mathbf{x}_p(t) = X(t)\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t)v_1(t) + y_2(t)v_2(t) \\ y_1'(t)v_1(t) + y_2'(t)v_2(t) \end{bmatrix}$$

First component is part. sol. of (8):

$$y_p(t) = y_1(t)v_1(t) + y_2(t)v_2(t)$$

where
$$\begin{cases} v_1(t) = -\int [y_2(t)F(t)/W(t)]\,dt \\ v_2(t) = \int [y_1(t)F(t)/W(t)]\,dt \end{cases}$$

Example

Ex.: Find particular solution of

$$y'' + y = \tan t$$

F.S.S. of $y'' + y = 0$:

$$\left.\begin{array}{rcl} y_1(t) & = & \cos t \\ y_2(t) & = & \sin t \end{array}\right\} W(t) = \cos^2 t + \sin^2 t = 1$$

$$\begin{array}{rcl} v_1(t) & = & -\int \sin t \tan t \, dt \\ & = & \sin t - \ln|\sec t + \tan t| \\ v_2(t) & = & \int \cos t \tan t \, dt = \cos t \end{array}$$

$$\Rightarrow y_p(t) = -\cos t \ln|\sec t + \tan t|$$