Math 3331 Differential Equations

5.1 Definition of the Laplace Transform

Blerina Xhabli

Department of Mathematics, University of Houston

blerina@math.uh.edu
math.uh.edu/∼blerina/teaching.html

5.1 Definition of the Laplace Transform

- Use of the Laplace Transform
  - DEs with constant coefficients
  - Discontinuous forcings
- Definition of the Laplace Transform
- Examples
  - $\mathcal{L}(1)(s) = 1/s$, $s > 0$
  - $\mathcal{L}(e^{at})(s) = 1/s - a$, $s > a$
  - $\mathcal{L}(t)(s) = 1/s^2$
  - $\mathcal{L}(t^n)(s) = n!/s^{n+1}$
  - $\mathcal{L}(\sin at)(s)$ and $\mathcal{L}(\cos at)(s)$
  - $\mathcal{L}(f)(s)$, f being discontinous
- Piecewise Continuous Functions
- Functions of Exponential Order
- Existence of the Laplace Transform

Use of the Laplace Transform

- Technique for solving linear DEs with constant coefficients
- Useful for discontinuous forcings

Differential Equation $\longrightarrow$ Laplace Transform $\longrightarrow$ Algebraic Equation

Solve Algebraic Equation $\longrightarrow$ Inverse Laplace Transform $\longrightarrow$ Solution

Definition of the Laplace Transform

Def.: Given a real or complex function $f(t)$, the Laplace ($\mathcal{L}$) transform of f is the following function of s:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)\,dt \\ &\equiv \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,dt \end{aligned}$$

Notation:

$$F(s) = \mathcal{L}(f)(s) = \mathcal{L}\{f(t)\}(s)$$

Example 1: $\mathcal{L}(1)(s) = 1/s$, $s > 0$

$$\mathcal{L}(1)(s) = \int_0^\infty 1\, e^{-st}\, dt = \lim_{T\to\infty} \int_0^T e^{-st}\, dt = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-st} \Big|_{t=0}^{T} \right]$$

$$= \lim_{T\to\infty} \left[-\frac{1}{s} e^{-sT} + \frac{1}{s} \right] = \frac{1}{s} \quad \text{for } s > 0$$

Example 2: $\mathcal{L}(e^{at})(s) = 1/s - a$, $s > a$

$$\mathcal{L}(e^{at})(s) = \int_0^\infty e^{at}\, e^{-st}\, dt = \lim_{T\to\infty} \int_0^T e^{-(s-a)t}\, dt$$

$$= \lim_{T\to\infty} \left[-\frac{1}{s-a} e^{-(s-a)t} \Big|_{t=0}^{T} \right]$$

$$= \lim_{T\to\infty} \left[-\frac{1}{s-a} e^{-(s-a)T} + \frac{1}{s-a} \right] = \frac{1}{s-a} \quad \text{for } s > a$$

Example 3: $\mathcal{L}(t)(s) = 1/s^2$

$$\mathcal{L}(t)(s) = \int_0^\infty t\, e^{-st}\, dt$$

$$= \lim_{T\to\infty} \int_0^T t\, e^{-st}\, dt$$

$$= \lim_{T\to\infty} \left(-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right) \Bigg|_{t=0}^{T}$$

$$= \lim_{T\to\infty} \left(-\frac{T}{s} e^{-sT} - \frac{1}{s^2} e^{-sT} + \frac{1}{s^2} \right) = \frac{1}{s^2}$$

Integration by parts

$$\int t\, e^{-st}\, dt$$

$$= -\frac{1}{s} \left(te^{-st} - \int e^{-st}\, dt \right)$$

$$= -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st}$$

Example 4: $\mathcal{L}(t^n)(s) = n!/s^{n+1}$

$$\mathcal{L}(t^n)(s) = \int_0^\infty t^n\, e^{-st}\, dt$$

$$= \lim_{T\to\infty} \int_0^T t^n\, e^{-st}\, dt$$

$$= \lim_{T\to\infty} \left(-\frac{t^n}{s} e^{-st} - \cdots - \frac{n!}{s^{n+1}} e^{-st} \right) \Bigg|_{t=0}^{T}$$

$$= \lim_{T\to\infty} \left(-\frac{T^n}{s} e^{-sT} - \cdots - \frac{n!}{s^{n+1}} e^{-sT} + \frac{n!}{s^{n+1}} \right)$$

$$= \frac{n!}{s^{n+1}}$$

Integration by parts

$$\int t^n\, e^{-st}\, dt$$

$$= -\frac{1}{s} t^n e^{-st}$$

$$+ \frac{n}{s} \int t^{n-1}\, e^{-st}\, dt$$

$$= \cdots$$

$$= -\frac{t^n}{s} e^{-st} - \cdots - \frac{n!}{s^{n+1}} e^{-st}$$

Example 5: $\mathcal{L}(\sin at)(s)$ and $\mathcal{L}(\cos at)(s)$

$$\mathcal{L}(\sin at)(s) = \int_0^{\infty} \sin at\, e^{-st}\, dt = \cdots = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}(\cos at)(s) = \int_0^{\infty} \cos at\, e^{-st}\, dt = \cdots \frac{s}{s^2 + a^2}$$

Example 6: $\mathcal{L}(f)(s)$, f being discontinous

Compute the Laplace transform of the step function

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$$

$$\mathcal{L}(f)(s) = \int_0^1 1\, e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_{t=0}^{1}$$

$$= -\frac{1}{s} e^{-s} + \frac{1}{s}.$$

Piecewise Continuous Functions

Def.: $f(t)$ is piecewise continuous if

- in any finite interval $0 < t < T$ there are at most finitely many discontinuities
- at any point of discontinuity t_d the left and right limits $f_{\mp}$ exist:

$$f_-(t_d) = \lim_{t \to t_d -} f(t), \; f_+(t_d) = \lim_{t \to t_d +} f(t)$$

Ex.: $f(t) = \begin{Bmatrix} 0 & \text{if} & 0 \le t < 1 \\ e^{t-1} & \text{if} & t \ge 1 \end{Bmatrix}$ has a discontinuity at $t_d = 1$:

$$f_-(1) = 0, \; f_+(1) = 1$$

Functions of Exponential Order

Def.: $f(t)$ is of exponential order if there are constants C, a s.t.

$$|f(t)| \leq Ce^{at} \text{ for all } t$$

Meaning: $f(t)$ grows at most exponentially if $t \longrightarrow \infty$

Ex.: e^{t^2} is *not* of exponential order

Ex.: $e^{10,000t}$ is of exponential order

Existence of the Laplace Transform

Thm.: If $f(t)$ is piecewise continuous in $0 \leq t < \infty$ and of exponential order, then $\mathcal{L}(f)(s)$ exists for $s > a$.