

Math 3331 Differential Equations

5.3 The Inverse Laplace Transform

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5.3 The Inverse Laplace Transform

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- Partial Fractions
 - Inverse \mathcal{L} -Transform of Rational Functions
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 - Examples



Uniqueness Theorem

Thm.: If $f(t)$ and $g(t)$ are piecewise continuous on $0 \leq t < \infty$ and $\mathcal{L}(f)(s) = \mathcal{L}(g)(s)$ for $s > a$, then $f(t) = g(t)$ for all t in $0 \leq t < \infty$ at which $f(t)$ is continuous.



\mathcal{L} -Transform Pairs

$$F(s) = \mathcal{L}(f)(s) = \mathcal{L}\{f(t)\}(s), \quad s > a$$

\mathcal{L} -transform pairs:

- $f(t)$ determines $F(s)$ uniquely in $s > a$
- $F(s)$ determines $f(t)$ uniquely in $0 \leq t < \infty$ except at discontinuity points.



Definition of the Inverse Laplace Transform

Def.: Given $F(s)$ and $f(t)$ s.t.
 $F(s) = \mathcal{L}(f)(s)$, then $f(t)$ is called
the inverse Laplace (\mathcal{L}) transform of
 $F(s)$, and is denoted by

$$f(t) = \mathcal{L}^{-1}(F)(t) = \mathcal{L}^{-1}\{F(s)\}(t)$$

$$F = \mathcal{L}(f) \quad \Leftrightarrow \quad f = \mathcal{L}^{-1}(F)$$



Table of Inverse \mathcal{L} -Transform

$$F = \mathcal{L}(f) \Leftrightarrow f = \mathcal{L}^{-1}(F)$$

$F(s)$	$\mathcal{L}^{-1}\{F(s)\}(t)$
$\frac{1}{s-c}$	e^{ct}
$\frac{1}{(s-c)^k}$	$\frac{t^{k-1}}{(k-1)!} e^{ct}$
$\frac{1}{(s-\alpha)^2 + \beta^2}$	$\frac{e^{\alpha t} \sin \beta t}{\beta}$
$\frac{s-\alpha}{(s-\alpha)^2 + \beta^2}$	$e^{\alpha t} \cos \beta t$



Exercise 5.3.2

2. Compute the inverse Laplace transform of $Y(s) = \frac{1}{3-5s}$.

2. Adjust as follows.

$$Y(s) = \frac{2}{3-5s} = -\frac{2}{5} \cdot \frac{1}{s-3/5}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ -\frac{2}{5} \cdot \frac{1}{s-3/5} \right\} \\ &= -\frac{2}{5} \mathcal{L}^{-1} \left\{ \frac{1}{s-3/5} \right\} \\ &= -\frac{2}{5} e^{(3/5)t} \end{aligned}$$



Exercise 5.3.4

4. Compute the inverse Laplace transform of $Y(s) = \frac{5s}{s^2+9}$

4. Adjust as follows.

$$Y(s) = \frac{5s}{s^2+9} = 5 \cdot \frac{s}{s^2+9}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ 5 \cdot \frac{s}{s^2+9} \right\} \\ &= 5 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} \\ &= 5 \cos 3t \end{aligned}$$



Exercise 5.3.6

6. Compute the inverse Laplace transform of $Y(s) = \frac{2}{3s^4}$

6. Adjust as follows.

$$Y(s) = \frac{2}{3s^4} = \frac{1}{9} \cdot \frac{3!}{s^4}$$

Thus, by linearity,

$$\begin{aligned}y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{9} \cdot \frac{3!}{s^4} \right\} \\&= \frac{1}{9} \mathcal{L}^{-1} \left\{ \frac{3!}{s^4} \right\} \\&= \frac{1}{9} t^3\end{aligned}$$



Exercise 5.3.7

7. Compute the inverse Laplace transform of $Y(s) = \frac{3s+2}{s^2+25}$

7. Adjust as follows:

$$\begin{aligned} Y(s) &= \frac{3s+2}{s^2+25} \\ &= \frac{3s}{s^2+25} + \frac{2}{s^2+25} \\ &= 3 \cdot \frac{s}{s^2+25} + \frac{2}{5} \cdot \frac{5}{s^2+25}. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ 3 \cdot \frac{s}{s^2+25} + \frac{2}{5} \cdot \frac{5}{s^2+25} \right\} \\ &= 3 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+25} \right\} + \frac{2}{5} \mathcal{L}^{-1} \left\{ \frac{5}{s^2+25} \right\} \\ &= 3 \cos 5t + \frac{2}{5} \sin 5t. \end{aligned}$$



Exercise 5.3.9

9. Compute the inverse Laplace transform of $Y(s) = \frac{1}{3-4s} + \frac{3-2s}{s^2+49}$

9. Adjust as follows:

$$\begin{aligned} Y(s) &= \frac{1}{3-4s} + \frac{3-2s}{s^2+49} \\ &= \frac{1}{-4} \cdot \frac{1}{s-3/4} + \frac{3}{s^2+49} - \frac{2s}{s^2+49} \\ &= -\frac{1}{4} \cdot \frac{1}{s-3/4} + \frac{3}{7} \cdot \frac{7}{s^2+49} \\ &\quad - 2 \cdot \frac{s}{s^2+49}. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ -\frac{1}{4} \cdot \frac{1}{s-3/4} + \frac{3}{7} \cdot \frac{7}{s^2+49} \right. \\ &\quad \left. - 2 \cdot \frac{s}{s^2+49} \right\} \\ &= -\frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{1}{s-3/4} \right\} + \frac{3}{7} \mathcal{L}^{-1} \left\{ \frac{7}{s^2+49} \right\} \\ &\quad - 2 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+49} \right\} \\ &= -\frac{1}{4} e^{(3/4)t} + \frac{3}{7} \sin 7t - 2 \cos 7t. \end{aligned}$$



Exercise 5.3.11

11. Compute the inverse Laplace transform of $Y(s) = \frac{5}{(s+2)^3}$

11. Note the transform pair:

$$t^2 \iff \frac{2}{s^3}$$

By Proposition 2.12,

$$e^{-2t}t^2 \iff \frac{2}{(s+2)^3}.$$

Thus,

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\left\{\frac{5}{(s+2)^3}\right\} \\&= \mathcal{L}^{-1}\left\{\frac{5}{2} \cdot \frac{2}{(s+2)^3}\right\} \\&= \frac{5}{2} \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^3}\right\} \\&= \frac{5}{2} e^{-2t} t^2.\end{aligned}$$



Exercise 5.3.14

14. Compute the inverse Laplace transform of $Y(s) = \frac{4(s-1)}{(s-1)^2+4}$

14. Note the transform pair.

$$\cos 2t \leftrightarrow \frac{s}{s^2 + 4}$$

By Proposition 2.12,

$$e^t \cos 2t \leftrightarrow \frac{s-1}{(s-1)^2+4}.$$

Hence,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{4(s-1)}{(s-1)^2+4} \right\} \\ &= 4 \mathcal{L}^{-1} \left\{ \frac{s-1}{(s-1)^2+4} \right\} \\ &= 4e^t \cos 2t. \end{aligned}$$



Exercise 5.3.15

15. Compute the inverse Laplace transform of $Y(s) = \frac{2s-3}{(s-1)^2+5}$

15. Note the transform pairs:

$$\cos \sqrt{5}t \iff \frac{s}{s^2+5}$$

$$\sin \sqrt{5}t \iff \frac{\sqrt{5}}{s^2+5}$$

By Proposition 2.12,

$$e^t \cos \sqrt{5}t \iff \frac{s-1}{(s-1)^2+5}$$

$$e^t \sin \sqrt{5}t \iff \frac{\sqrt{5}}{(s-1)^2+5}.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{2s-3}{(s-1)^2+5} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{2s-2}{(s-1)^2+5} - \frac{1}{(s-1)^2+5} \right\} \\ &= \mathcal{L}^{-1} \left\{ 2 \cdot \frac{s-1}{(s-1)^2+5} \right. \\ &\quad \left. - \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{(s-1)^2+5} \right\} \\ &= 2 \mathcal{L}^{-1} \left\{ \frac{s-1}{(s-1)^2+5} \right\} \\ &\quad - \frac{1}{\sqrt{5}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{5}}{(s-1)^2+5} \right\} \\ &= 2e^t \cos \sqrt{5}t - \frac{1}{\sqrt{5}} e^t \sin \sqrt{5}t \\ &= e^t (2 \cos \sqrt{5}t - \frac{\sqrt{5}}{5} \sin \sqrt{5}t). \end{aligned}$$

Exercise 5.3.17

17. Compute the inverse Laplace transform of $Y(s) = \frac{3s+2}{s^2+4s+29}$

17. Complete the square.

$$Y(s) = \frac{3s+2}{s^2+4s+29} = \frac{3s+2}{(s+2)^2+25}$$

Note the transform pairs.

$$\cos 5t \iff \frac{s}{s^2+25}$$

$$\sin 5t \iff \frac{5}{s^2+25}$$

By Proposition 2.12,

$$e^{-2t} \cos 5t \iff \frac{s+2}{(s+2)^2+25}$$

$$e^{-2t} \sin 5t \iff \frac{5}{(s+2)^2+25}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{3s+2}{(s+2)^2+25} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{3s+6}{(s+2)^2+25} - \frac{4}{(s+2)^2+25} \right\} \\ &= \mathcal{L}^{-1} \left\{ 3 \cdot \frac{s+2}{(s+2)^2+25} - \frac{4}{5} \cdot \frac{5}{(s+2)^2+25} \right\} \\ &= 3 \mathcal{L}^{-1} \left\{ \frac{s+2}{(s+2)^2+25} \right\} - \frac{4}{5} \mathcal{L}^{-1} \left\{ \frac{5}{(s+2)^2+25} \right\} \\ &= 3e^{-2t} \cos 5t - \frac{4}{5} e^{-2t} \sin 5t \\ &= e^{-2t} \left(3 \cos 5t - \frac{4}{5} \sin 5t \right). \end{aligned}$$

Inverse \mathcal{L} -Transform of Rational Functions

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Form: $F(s) = \frac{P(s)}{Q(s)}$

- $P(s)$, $Q(s)$: polynomials
- degree of $P <$ degree of Q

Assume $Q(s)$ has k **distinct roots**

Partial Fraction Decomposition (PFD):

$$F(s) = \sum_{\{\lambda\}} F_{\lambda}(s)$$

$F_{\lambda}(s)$: contribution from root λ

Linearity \Rightarrow

$$\mathcal{L}^{-1}(F)(t) = \sum_{\{\lambda\}} \mathcal{L}^{-1}(F_{\lambda})(t)$$

Let m be the multiplicity of λ . Set $Q_{\lambda}(s) = Q(s)/(s - \lambda)^m \Rightarrow Q_{\lambda}(\lambda) \neq 0$



Simple Root: ($m = 1$)

Simple Root: ($m = 1$)

$$F_\lambda(s) = \frac{A}{s - \lambda}, \quad A = \frac{P(\lambda)}{Q_\lambda(\lambda)}$$

$$\Rightarrow \mathcal{L}^{-1}(F_\lambda)(t) = Ae^{\lambda t}$$

Complex Case: Assume $\lambda = \alpha + i\beta$,
 $\bar{\lambda} = \alpha - i\beta$ are a complex conjugate
 pair of simple roots

$$\Rightarrow F_\lambda(s) + F_{\bar{\lambda}}(s) = \frac{A}{s - \lambda} + \frac{\bar{A}}{s - \bar{\lambda}}$$

$$\Rightarrow \mathcal{L}^{-1}(F_\lambda + F_{\bar{\lambda}})(t) = Ae^{\lambda t} + \bar{A}e^{\bar{\lambda}t}$$

$$= 2\operatorname{Re}(Ae^{\lambda t})$$

Real version: let $A = a + ib$

$$\Rightarrow F_\lambda(s) + F_{\bar{\lambda}}(s) = \frac{2a(s - \alpha) - 2b\beta}{(s - \alpha)^2 + \beta^2}$$

$$\Rightarrow \mathcal{L}^{-1}(F_\lambda + F_{\bar{\lambda}})(t) =$$

$$2e^{\alpha t}(a \cos \beta t - b \sin \beta t)$$



Multiple Root: ($m > 1$)

Multiple Root: ($m > 1$)

$$F_\lambda(s) = \frac{A_m}{s-\lambda} + \frac{A_{m-1}}{(s-\lambda)^2} + \dots \\ + \frac{A_2}{(s-\lambda)^{m-1}} + \frac{A_1}{(s-\lambda)^m}$$

$$\Rightarrow \mathcal{L}^{-1}(F_\lambda)(s) = e^{\lambda t} [A_m + A_{m-1}t + \dots \\ + A_1 t^{m-1} / (m-1)!]$$

For multiple complex pairs $\lambda, \bar{\lambda}$:

$$\mathcal{L}^{-1}(F_\lambda + F_{\bar{\lambda}})(t) = \\ 2 \left[\operatorname{Re}(A_m e^{\lambda t}) + t \operatorname{Re}(A_{m-1} e^{\lambda t}) + \dots \right. \\ \left. + \frac{t^{m-2} \operatorname{Re}(A_2 t e^{\lambda t})}{(m-2)!} + \frac{t^{m-1} \operatorname{Re}(A_1 t e^{\lambda t})}{(m-1)!} \right]$$

Coefficients:

$$A_j = \frac{1}{(j-1)!} \left[\frac{d^{j-1}}{ds^{j-1}} \left(\frac{P(s)}{Q_\lambda(s)} \right) \right]_{s=\lambda}$$

For $m = 2$:

$$A_1 = \frac{P(\lambda)}{Q_\lambda(\lambda)}, \quad A_2 = \left[\frac{d}{ds} \left(\frac{P(s)}{Q_\lambda(s)} \right) \right]_{s=\lambda}$$



Example 1

Ex. 1: $F(s) = \frac{s+9}{s^2-2s-3} = \frac{s+9}{(s+1)(s-3)}$

Roots: $\lambda_1 = -1, \lambda_2 = 3 \rightarrow$

$$F(s) = F_{-1}(s) + F_3(s)$$

$$F_{-1}(s) = \frac{A}{s+1}, \quad F_3(s) = \frac{B}{s-3}$$

$$Q_{-1}(s) = \frac{(s+1)(s-3)}{s+1} = s-3$$

$$\Rightarrow A = \left. \frac{s+9}{s-3} \right|_{s=-1} = -2$$

$$Q_3(s) = \frac{(s+1)(s-3)}{s-3} = s+1$$

$$\Rightarrow B = \left. \frac{s+9}{s+1} \right|_{s=3} = 3$$

$$\Rightarrow F(s) = \frac{-2}{s+1} + \frac{3}{s-3}$$

$$\Rightarrow \mathcal{L}^{-1}(t) = -2e^{-t} + 3e^{3t}$$



Example 1(cont.)

Other methods for finding A, B :

(see text, Sec. 5.3, Example 3.6)

$$\frac{s+9}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3}$$

$$\Rightarrow s+9 = A(s-3) + B(s+1) \quad (2)$$

Substitution method:

Substitute two values for s in (2):

$$s = 3 \Rightarrow 12 = 4B \Rightarrow B = 3$$

$$s = -1 \Rightarrow 8 = -4A \Rightarrow A = -2$$

Coefficient method: Rewrite (2) as

$$s+9 = (A+B)s + (-3A+B)$$

Equate coefficients of powers of s :

$$\Rightarrow \begin{cases} 1 = A+B \\ 9 = -3A+B \end{cases}$$

$$\Rightarrow \begin{cases} A = -2 \\ B = 3 \end{cases}$$



Example 2

Ex. 2:

$$\begin{aligned} Y(s) &= \frac{s-2}{s^2-2s-3} = \frac{s-2}{(s+1)(s-3)} \\ &= \frac{A}{s+1} + \frac{B}{s-3} \end{aligned}$$

$$A = \left. \frac{s-2}{s-3} \right|_{s=-1} = \frac{3}{4}$$

$$B = \left. \frac{s-2}{s+1} \right|_{s=3} = \frac{1}{4}$$

$$\Rightarrow Y(s) = \frac{1}{4} \left(\frac{3}{s+1} + \frac{1}{s-3} \right)$$

$$\Rightarrow \mathcal{L}^{-1}(Y)(t) = \frac{1}{4} (3e^{-t} + e^{3t})$$



Example 3

Ex. 3: $F(s) = \frac{1}{s^2+4s+13} = \frac{1}{(s+2)^2+9}$

This is of the form

$$\frac{1}{(s - \alpha)^2 + \beta^2} \quad (\alpha = -2, \beta = 3)$$

with inverse transform (see table)

$$(1/\beta)e^{\alpha t} \sin \beta t$$

$$\Rightarrow \mathcal{L}^{-1}(F)(t) = (1/3)e^{-2t} \sin 3t$$

See text, Sec. 5.3, Example 3.6, for coefficient and substitution methods.



Example 4

Ex. 4: $F(s) = \frac{2s^2 + s + 13}{(s-1)[(s+1)^2 + 4]}$

(see text, Sec. 5.3, Example 3.9)

$$(s+1)^2 + 4 = (s+1+2i)(s+1-2i)$$

\Rightarrow roots of $Q(s)$:

$$\lambda_1 = 1, \lambda_2 = -1 + 2i, \lambda_3 = \overline{\lambda_2}$$

$$F_{\lambda_1}(s) = \frac{A}{s-1}, \quad A = \left. \frac{2s^2 + s + 13}{(s+1)^2 + 4} \right|_{s=1} = 2$$

$$\Rightarrow \mathcal{L}^{-1}(F_{\lambda_1})(t) = 2e^t$$

Work on λ_2 : $F_{\lambda_2}(s) = \frac{B}{s+1-2i}$

$$\begin{aligned} B &= \left. \frac{2s^2 + s + 13}{(s-1)(s+1+2i)} \right|_{s=-1+2i} \\ &= \frac{2(1-4i-4) + (-1+2i) + 13}{(-2+2i)4i} \\ &= \frac{6-6i}{-8-8i} = -\frac{3}{4} \frac{1-i}{1+i} = \frac{3i}{4} \\ \Rightarrow F_{\lambda_2}(s) + F_{\lambda_2}(s) &= \frac{3}{4} \left(\frac{i}{s+1-2i} - \frac{i}{s+1+2i} \right) \\ &= \frac{-3}{(s+1)^2 + 4} \\ \Rightarrow \mathcal{L}^{-1}(F_{\lambda_2} + F_{\lambda_2})(t) &= -\frac{3}{2} e^{-t} \sin 2t \\ \Rightarrow \mathcal{L}^{-1}(F)(t) &= 2e^t - (3/2)e^{-t} \sin 2t \end{aligned}$$



Example 5

Ex. 5: $Y(s) = \frac{s^2+s+4}{(s^2+1)(s^2+4)}$

$$\left. \begin{aligned} s^2 + 1 &= (s - i)(s + i) \\ s^2 + 4 &= (s - 2i)(s + 2i) \end{aligned} \right\} \Rightarrow \text{roots:}$$

$$\lambda_1 = i, \lambda_2 = -i, \lambda_3 = 2i, \lambda_4 = -2i$$

$$Y_{\lambda_1}(s) = \frac{A}{s - i}$$

$$\begin{aligned} A &= \left. \frac{s^2 + s + 4}{(s + i)(s^2 + 4)} \right|_{s=i} \\ &= \frac{3 + i}{6i} = \frac{1}{6}(1 - 3i) \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathcal{L}^{-1}(Y_{\lambda_1} + Y_{\bar{\lambda}_1})(t) &= 2\text{Re}(Ae^{it}) \\ &= \frac{1}{3}(\cos t + 3 \sin t) \end{aligned}$$

$$Y_{\lambda_3}(s) = \frac{B}{s - 2i}$$

$$\begin{aligned} A &= \left. \frac{s^2 + s + 4}{(s^2 + 1)(s + 2i)} \right|_{s=2i} \\ &= \frac{2i}{(-3)4i} = -\frac{1}{6} \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathcal{L}^{-1}(Y_{\lambda_3} + Y_{\bar{\lambda}_3})(t) &= 2\text{Re}(Be^{2it}) \\ &= -\frac{1}{3} \cos 2t \end{aligned}$$

$$\mathcal{L}^{-1}(Y)(t) = (1/3)(\cos t + 3 \sin t - \cos 2t)$$



Example 6

Ex. 6: $Y(s) = \frac{1}{(s+1)(s-1)^2}$

Roots: $\lambda_1 = -1$, $\lambda_2 = 1$ ($m = 2$)

$$Y_{-1}(s) = \frac{A}{s+1}, \quad A = \frac{1}{(s-1)^2} \Big|_{s=-1} = \frac{1}{4}$$

$$Y_1(s) = \frac{B_1}{(s-1)^2} + \frac{B_2}{s-1}$$

$$B_1 = \frac{1}{s+1} \Big|_{s=1} = \frac{1}{2}$$

$$B_2 = \left(\frac{d}{ds} \frac{1}{s+1} \right) \Big|_{s=1} = -\frac{1}{4}$$

$$Y(s) = \frac{1}{4} \frac{1}{s+1} - \frac{1}{4} \frac{1}{s-1} + \frac{1}{2} \frac{1}{(s-1)^2}$$

$$\mathcal{L}^{-1}(Y)(t) = \frac{1}{4}(e^{-t} - e^t + 2te^t)$$



Example 7

Ex. 7: $Y(s) = \frac{s}{(s^2+2s+2)(s^2+4)}$

$Q(s) = [(s+1)^2 + 1](s^2 + 4)$: factorize $(s+1)^2 + 1 = (s+1-i)(s+1+i)$,
 $s^2 + 4 = (s-2i)(s+2i) \Rightarrow$ roots $\lambda_1 = -1+i$, $\lambda_2 = \bar{\lambda}_1$, $\lambda_3 = 2i$, $\lambda_4 = \bar{\lambda}_3$

$$Y_{\lambda_1}(s) = \frac{A}{s+1-i}, \quad A = \frac{s}{(s+1+i)(s^2+4)} \Big|_{s=-1+i} = \frac{-1+i}{2i((1-i)^2+4)}$$

$$= \frac{-1+i}{2i(4-2i)} = \frac{1}{4} \frac{-1+i}{1+2i} = \frac{1}{4} \frac{1}{5} (-1+i)(1-2i) = \frac{1}{20}(1+3i)$$

$$\Rightarrow \mathcal{L}^{-1}(Y_{\lambda_1} + Y_{\bar{\lambda}_1})(t) = 2e^{-t} \operatorname{Re}\left(\frac{1}{20}(1+3i)e^{it}\right) = \frac{1}{10}e^{-t}(\cos t - 3 \sin t)$$

$$Y_{\lambda_3}(s) = \frac{B}{s-2i}, \quad B = \frac{s}{(s^2+2s+2)(s+2i)} \Big|_{s=2i} = \frac{2i}{(-2+4i)4i}$$

$$= -\frac{1}{4} \frac{1}{1-2i} = -\frac{1}{20}(1+2i)$$

$$\Rightarrow \mathcal{L}^{-1}(Y_{\lambda_3} + Y_{\bar{\lambda}_3})(t) = 2 \operatorname{Re}\left(-\frac{1}{20}(1+2i)e^{2it}\right) = -\frac{1}{10}(\cos 2t - 2 \sin 2t)$$

$$\Rightarrow \mathcal{L}^{-1}(Y)(t) = \frac{1}{10}(e^{-t} \cos t - 3e^{-t} \sin t - \cos 2t + 2 \sin 2t)$$

