

# Math 3331 Differential Equations

## 5.4 Using the $\mathcal{L}$ -Transform to Solve ODE

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## 5.4 Using the $\mathcal{L}$ -Transform to Solve ODE

- Basic Idea
- Examples



# Basic Idea

$$\text{Basic Idea: } \left\{ \begin{array}{l} \text{IVP} \\ \text{for } y(t) : \\ \text{ODE+IC} \end{array} \right\} \xrightarrow{\mathcal{L}} \left\{ \begin{array}{l} \text{algebraic} \\ \text{equation} \\ \text{for } Y(s) \end{array} \right\} \xrightarrow{\text{solve}} Y(s) \xrightarrow{\mathcal{L}^{-1}} y(t)$$



# Example 8

**Ex. 8:**  $y'' + y = \cos 2t$

$$y(0) = 0, y'(0) = 1$$

$\mathcal{L}$ -transform ODE:

$$\mathcal{L}(y'' + y) = \mathcal{L}\{\cos 2t\}$$

$$\begin{aligned}\mathcal{L}(y'') &= s^2Y - sy(0) - y'(0) \\ &= s^2Y - 1\end{aligned}$$

$$\mathcal{L}(y) = Y$$

$$\Rightarrow \mathcal{L}(y'' + y) = (s^2 + 1)Y - 1$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2 + 4}$$

$$\Rightarrow (s^2 + 1)Y - 1 = \frac{s}{s^2 + 4}$$

$$\begin{aligned}\Rightarrow Y(s) &= \frac{1}{s^2 + 1} + \frac{s}{(s^2 + 1)(s^2 + 4)} \\ &= \frac{s^2 + s + 4}{(s^2 + 1)(s^2 + 4)}\end{aligned}$$

$$y(t) = \mathcal{L}^{-1}(Y)(t). \text{ From Ex. 5 } \Rightarrow$$

$$y(t) = \frac{1}{3}(\cos t + 3 \sin t - \cos 2t)$$



# Example 9

**Ex. 9:**  $y'' - 2y' - 3y = 0$

$$y(0) = 1, y'(0) = 0$$

$$\mathcal{L}(y'') = s^2Y - s$$

$$\mathcal{L}(y') = sY - 1$$

$$\Rightarrow \mathcal{L}(y'' - 2y' - 3y) = (s^2 - 2s - 3)Y - s + 2 = 0$$

$$\Rightarrow Y(s) = \frac{s - 2}{s^2 - 2s - 3}, \quad y(t) = \mathcal{L}^{-1}(Y)(t)$$

From Ex. 2:  $y(t) = (1/4)(e^{3t} + 3e^{-t})$



# Example 10

**Ex. 10:**  $y'' - y = e^t$ ,  $y(0) = y'(0) = 0$

$$\begin{aligned} \mathcal{L}(y'' - y) &= (s^2 - 1)Y, \quad \mathcal{L}\{e^t\} = \frac{1}{s - 1} \\ \Rightarrow (s^2 - 1)Y &= 1/(s - 1) \\ \Rightarrow Y(s) &= \frac{1}{(s^2 - 1)(s - 1)} = \frac{1}{(s + 1)(s - 1)^2} \\ \text{Ex. 6} \Rightarrow y(t) &= (e^{-t} - e^t + 2te^t)/4 \end{aligned}$$



# Example 11

**Ex. 11:**  $y'' + 2y' + 2y = \cos 2t$ ,  $y(0) = 0$ ,  $y'(0) = 1$

$$\mathcal{L}(y'' + 2y' + 2y) = (s^2Y - 1) + 2(sY) + 2Y = (s^2 + 2s + 2)Y - 1$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2 + 4} \Rightarrow (s^2 + 2s + 2)Y - 1 = \frac{s}{s^2 + 4}$$

$$\Rightarrow Y(s) = \frac{1}{s^2 + 2s + 2} + \frac{s}{(s^2 + 2s + 2)(s^2 + 4)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2s + 2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s + 1)^2 + 1}\right\} = e^{-t} \sin t$$

$$\begin{aligned} \text{From Ex. 7: } \mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 2s + 2)(s^2 + 4)}\right\} \\ = \frac{1}{10}(e^{-t} \cos t - 3e^{-t} \sin t - \cos 2t + 2 \sin 2t) \end{aligned}$$

$$\Rightarrow y(t) = \frac{1}{10}(e^{-t} \cos t + 7e^{-t} \sin t - \cos 2t + 2 \sin 2t)$$

