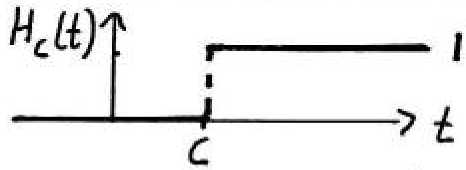
5.5 Discontinuous Forcing Functions

Heaviside step function ($c \geq 0$)

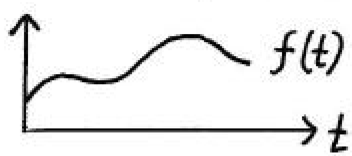


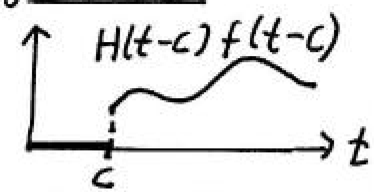
$$H_c(t) \equiv H(t-c) = \begin{cases} 1 & \text{for } t \geq c \\ 0 & \text{for } t < c \end{cases}$$

Since we restrict to $t \geq 0$, identify

$$H_0(t) \equiv H(t) = 1 \quad (\text{in } t \geq 0)$$

t-shift property of $\mathcal{L}$:





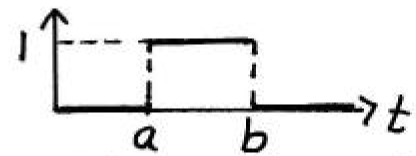
$$\mathcal{L}\{H(t-c)\,f(t-c)\} = e^{-cs}F(s)$$

or $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = H(t-c)\,f(t-c)$

$$\Rightarrow \mathcal{L}\{H_c(t)\} = e^{-cs}\mathcal{L}(1) = \frac{e^{-cs}}{s}$$

Boxcar (Interval) function

$$H_{ab}(t) = H_a(t) - H_b(t) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } a \leq t < b \\ 0 & \text{for } t \geq b \end{cases}$$



$(a < b)$

$$\mathcal{L}\{H_{ab}(t)\}(s) = (e^{-as} - e^{-bs})/s$$

Piecewise Defined Functions

Ex. 1: $g(t) = \begin{cases} 2t & \text{for } 0 \leq t < 1 \\ 2 & \text{for } 1 \leq t < \infty \end{cases}$

Use boxcars:

$$g(t) = 2t\,[1 - H(t-1)] + 2H(t-1)$$
$$= 2t - 2H(t-1)(t-1)$$

$$\Rightarrow \mathcal{L}(g)(s) = 2\mathcal{L}\{t\}(s) - 2\mathcal{L}\{H(t-1)(t-1)\}$$
$$= 2/s^2 - 2e^{-s}\mathcal{L}\{t\}(s)$$
$$= 2/s^2 - 2e^{-s}/s^2$$

Inverse $\mathcal{L}$-Transforms of Functions with Exponential Terms

Ex. 2 : $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s^2(s^2+1)}\right\}(t)$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}(t-1)$$

$$F(s) \equiv \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\{e^{-s}F(s)\}(t)$$

$$= \mathcal{L}^{-1}\{1/s^2\}(t-1) + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t-1)$$

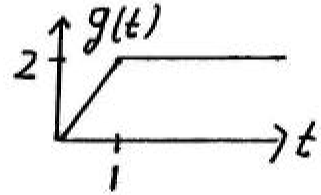
$$= H(t-1)[t-1-\sin(t-1)]$$

IVP's with Discontinuous Forcings

Ex. 3 : $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$

$$g(t) = \begin{cases} 2t & \text{for } 0 \le t < 1 \\ 2 & \text{for } t \ge 1 \end{cases}$$

(as in Ex. 1)



$$\mathcal{L}(y'' + y) = (s^2+1)Y - 1$$

$$\mathcal{L}(g) = 2(1-e^{-s})/s^2 \quad \text{(from Ex. 1)}$$

$$\Rightarrow (s^2+1)Y - 1 = 2(1-e^{-s})/s^2$$

$$\Rightarrow Y(s) = \frac{1}{s^2+1} + \frac{2}{s^2(s^2+1)} - \frac{2e^{-s}}{s^2(s^2+1)}$$

$$\Rightarrow y(t) = \sin t + 2(t - \sin t)$$

$$-2H(t-1)[t-1-\sin(t-1)] \quad \text{(Ex. 2)}$$

$$\Rightarrow y(t) = -\sin t + 2t[1 - H(t-1)] + 2H(t-1)[1 + \sin(t-1)]$$

$$= (2t - \sin t)[1 - H(t-1)] + [2 + 2\sin(t-1) - \sin t]H(t-1)$$

$$= \begin{cases} 2t - \sin t & \text{for } 0 \le t < 1 \\ 2 + 2\sin(t-1) - \sin t & \text{for } t \ge 1 \end{cases}$$

(see text, p. 221, for graph)