

Math 3331 Differential Equations

8.1 Introduction to Systems

Blerina Xhabli

Department of Mathematics, University of Houston

`blerina@math.uh.edu`
`math.uh.edu/~blerina/teaching.html`



8.1 Introduction to Systems Definitions and Examples

- Definitions
 - System of First Order ODEs
 - IVP: Existence and Uniqueness of Solution
- Examples
- Reduction of Higher Order Equations
- Worked out Examples from Exercises:
 - 1, 2, 7



System of First Order ODEs

System of 1st order ODEs:

$$x'_1 = f_1(t, x_1, \dots, x_n)$$

$$\vdots$$

$$x'_n = f_n(t, x_1, \dots, x_n)$$

Vector notation:

$$\mathbf{x} = [x_1, \dots, x_n]^T$$

$$\mathbf{f} = [f_1, \dots, f_n]^T$$

$$\mathbf{x}' = [x'_1, \dots, x'_n]^T$$

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad (1)$$

n : dimension of system

$n = 2$: planar system

- (1) is autonomous if \mathbf{f} does not depend on t
- (1) is non-autonomous if \mathbf{f} depends on t



IVP: Existence and Uniqueness of Solution

Initial Value Problem:

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (2)$$

Thm.: If \mathbf{f} is continuous in a region R and has continuous partial derivatives $\partial f_i / \partial x_j$ in R , (2) has a unique solution in R .



Example 1

$$\begin{aligned}\text{Ex.::} \quad x'_1 &= -ax_1x_2 \\ x'_2 &= ax_1x_2 - bx_2 \\ x'_3 &= bx_2\end{aligned}$$

$$\mathbf{x} = [x_1, x_2, x_3]^T$$

$$\mathbf{f}(\mathbf{x}) = [-ax_1x_2, ax_1x_2 - bx_2, bx_2]^T$$

$\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is 3d autonomous system



Example 2

$$\begin{aligned}\mathbf{Ex.}: \quad x' &= v \\ v' &= -x - 0.2v + 2 \cos t\end{aligned}$$

is $2d$ non-autonomous system



Reduction of Higher Order Equations

Thm. Any system of higher order ODEs in explicit form can be transformed to a (larger) system of first order ODEs.



Example 3

$$\text{Ex.: } x''' + xx'' = \cos t \quad (3)$$

Set $x_1 = x$, $x_2 = x'$, $x_3 = x''$

$$\begin{aligned} \Rightarrow x_1' &= x' = x_2 \\ x_2' &= x'' = x_3 \\ x_3' &= x''' = -xx'' + \cos t \\ &= -x_1x_3 + \cos t \end{aligned}$$

Hence equivalent system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -x_1x_3 + \cos t \end{aligned} \quad (4)$$

Given a solution $x(t)$ of (3) \Rightarrow
 $[x(t), x'(t), x''(t)]^T$ is solution of (4)

Conversely: Given a solution
 $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T$ of (4) \Rightarrow
 $x(t) = x_1(t)$ is a solution of (3)



General Higher Order ODEs

General Higher Order ODEs:
 n th order ODE in explicit form:

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$

Set $x_1 = x, x_2 = x', \dots, x_n = x^{(n-1)}$

\Rightarrow equivalent system:

$$\Rightarrow x'_1 = x' = x_2$$

$$x'_2 = x'' = x_3$$

$$\vdots$$

$$x'_{n-1} = x^{(n-1)} = x_n$$

$$x'_n = x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$

$$= f(t, x_1, x_2, \dots, x_n)$$

$$x'_1 = x_2$$

$$x'_2 = x_3$$

$$\vdots$$

$$x'_{n-1} = x_n$$

$$x'_n = f(t, x_1, x_2, \dots, x_n)$$



Exercise 8.1.1

Ex. 1: Is the system autonomous? What is the dimension?

$$\left. \begin{aligned} x' &= v \\ v' &= -x - 0.02v + 2 \cos t \end{aligned} \right\} \text{ is non-autonomous (} \cos t \text{). Dimension: 2}$$



Exercise 8.1.2

Ex. 2: Same questions as in Ex. 1

$$\left. \begin{aligned} \theta' &= \omega \\ \omega' &= -(g/L) \sin \theta + (k/m)\omega \end{aligned} \right\} \text{ is autonomous. Dimension: 2}$$



Exercise 8.1.7

Ex. 7: Show that given functions are solutions of initial value problem

$$\text{IVP: } \left\{ \begin{array}{l} x' = -4x + 6y \\ y' = -3x + 5y \end{array} \right\}, \left\{ \begin{array}{l} x(0) = 0 \\ y(0) = 1 \end{array} \right\}; \text{ functions } \left\{ \begin{array}{l} x(t) = 2e^{2t} - 2e^{-t} \\ y(t) = -e^{-t} + 2e^{2t} \end{array} \right\}$$

$$x'(t) = 4e^{2t} + 2e^{-t}, \quad -4x(t) + 6y(t) = -4(2e^{2t} - 2e^{-t}) + 6(-e^{-t} + 2e^{2t}) = 4e^{2t} + 2e^{-t}$$

$$y'(t) = e^{-t} + 4e^{2t}, \quad -3x(t) + 5y(t) = -3(2e^{2t} - 2e^{-t}) + 5(-e^{-t} + 2e^{2t}) = e^{-t} + 4e^{2t}$$

$$\text{IC: } x(0) = 0, \quad y(0) = 1,$$

hence $x(t), y(t)$ are solutions of IVP

