

Math 3331 Differential Equations

9.1 Constant Coefficients Linear Systems

Overview of Technique

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9.1 Constant Coefficients Linear Systems: Overview of Technique

- Homogeneous Systems
 - Eigenvalues and Eigenvectors
 - Solution of Homogeneous System
- Finding Eigenvalues
 - Characteristic Polynomials
 - Finding Eigenvalues and Eigenvectors
 - Distinct Eigenvalues and Independent Eigenvectors
- Fundamental Set of Solutions
- Complications
- Worked out Examples from Exercises:
 - 5, 11, 27, 29, 36, 39, 49, 51



Homogeneous System

Homogeneous system:

$$\mathbf{x}' = A\mathbf{x} \quad (1)$$

A : constant $n \times n$ -matrix

If $n = 1$: $x' = ax \Rightarrow x(t) = Ce^{at}$

Try exponential form for (1):

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{v} \quad (\mathbf{v} : \text{constant vector})$$

Sub $\mathbf{x}(t)$ in (1) \Rightarrow

$$\begin{aligned} \mathbf{x}'(t) &= \lambda e^{\lambda t} \mathbf{v} = A\mathbf{x}(t) = Ae^{\lambda t} \mathbf{v} \\ &\Rightarrow \lambda \mathbf{v} = A\mathbf{v} \end{aligned}$$



Eigenvalues and Eigenvectors

Def.: A number λ is an eigenvalue of A if there is a vector $\mathbf{v} \neq \mathbf{0}$ such that

$$A\mathbf{v} = \lambda\mathbf{v} \quad (2)$$

If λ is an eigenvalue, then any $\mathbf{v} \neq \mathbf{0}$ satisfying (2) is called an eigenvector for λ .



Solution of Homogeneous System

Homogeneous system:

$$\mathbf{x}' = A\mathbf{x} \quad (1)$$

A : constant $n \times n$ -matrix

Thm.: If λ is eigenvalue of A and \mathbf{v} is eigenvector for λ , then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution of (1).



Characteristic Polynomials

Def.: A number λ is an eigenvalue of A if there is a vector $\mathbf{v} \neq \mathbf{0}$ such that

$$A\mathbf{v} = \lambda\mathbf{v} \quad (2)$$

If λ is an eigenvalue, then any $\mathbf{v} \neq \mathbf{0}$ satisfying (2) is called an eigenvector for λ .

Rewrite (2) (using $\mathbf{v} = I\mathbf{v}$):

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

Since $\mathbf{v} \neq \mathbf{0} \Rightarrow \det(A - \lambda I) = 0$

Def.: $p(\lambda) = \det(A - \lambda I)$
= characteristic polynomial

Note: the degree of $p(\lambda)$ is n .
 $\Rightarrow p(\lambda)$ has n roots
(if counted with multiplicities)



Finding Eigenvalues and Eigenvectors

Thm.: The eigenvalues of A are the roots of

$$p(\lambda) = \det(A - \lambda I) = 0 \quad (3)$$

If λ is a root of (3), then any $\mathbf{v} \neq \mathbf{0}$ in $\text{null}(A - \lambda I)$ is an eigenvector for λ .

Def.: If λ is an eigenvalue of A , then $\text{null}(A - \lambda I)$ is called the eigenspace of λ .



Distinct Eigenvalues and Independent Eigenvectors

Thm.: Eigenvectors for distinct eigenvalues are linearly independent.



Fundamental Set of Solutions

Consequence:

If $p(\lambda)$ has n distinct real roots

$$\lambda_1, \dots, \lambda_n$$

then A has n linearly independent eigenvectors

$$\mathbf{v}_1, \dots, \mathbf{v}_n$$

$$\Rightarrow e^{\lambda_1 t} \mathbf{v}_1, \dots, e^{\lambda_n t} \mathbf{v}_n$$

is fundamental set of solutions.



Complications

Complications

- complex eigenvalues
- repeated roots



Exercise 9.1.5

Ex. 9.1.5: Find $p(\lambda)$ and eigenvalues “by hand” for $A = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}$

This is a 2×2 -matrix with $T = 1$, $D = -2 \Rightarrow p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$
 \Rightarrow Eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$.



Exercise 9.1.11

Ex. 9.1.11: Find $p(\lambda)$ and eigenvalues “by hand” for $A = \begin{bmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{bmatrix}$

$$\begin{aligned}
 p(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} -1 - \lambda & -4 & -2 \\ 0 & 1 - \lambda & 1 \\ -6 & -12 & 2 - \lambda \end{vmatrix} \\
 &= (-1)^{2+2}(1 - \lambda) \begin{vmatrix} -1 - \lambda & -2 \\ -6 & 2 - \lambda \end{vmatrix} + (-1)^{2+3}1 \begin{vmatrix} -1 - \lambda & -4 \\ -6 & -12 \end{vmatrix} \\
 &= (1 - \lambda)[(-1 - \lambda)(2 - \lambda) - 12] - [(-1 - \lambda)(-12) - 24] \\
 &= -(1 - \lambda)(-\lambda^2 + \lambda + 14) + 12(1 - \lambda) \\
 &= (1 - \lambda)(\lambda^2 - \lambda - 2) \\
 &= (1 - \lambda)(\lambda + 1)(\lambda - 2)
 \end{aligned}$$

\Rightarrow Eigenvalues 1, -1, 2



Exercise 9.1.27

Ex. 9.1.27: Find fundamental solution set “by hand” for $y' = Ay$ if

$$A = \begin{bmatrix} -3 & 0 & 2 \\ 6 & 3 & -12 \\ 2 & 2 & -6 \end{bmatrix}$$

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} -3 - \lambda & 0 & 2 \\ 6 & 3 - \lambda & -12 \\ 2 & 2 & -6 - \lambda \end{vmatrix} \\ &= (-1)^{1+1}(-3 - \lambda) \begin{vmatrix} 3 - \lambda & -12 \\ 2 & -6 - \lambda \end{vmatrix} + (-1)^{1+3}2 \begin{vmatrix} 6 & 3 - \lambda \\ 2 & 2 \end{vmatrix} \\ &= -(3 + \lambda)[(3 - \lambda)(-6 - \lambda) + 24] + 2[12 - 2(3 - \lambda)] \\ &= -(3 + \lambda)(\lambda^2 + 3\lambda + 6) + 4(\lambda + 3) = -(\lambda + 3)(\lambda^2 + 3\lambda + 2) \\ &= -(\lambda + 3)(\lambda + 1)(\lambda + 2) \end{aligned}$$

\Rightarrow eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$. Find eigenvectors:



Exercise 9.1.27 (cont.)

1. $\lambda_1 = -1$:

$$A + I = \begin{bmatrix} -2 & 0 & 2 \\ 6 & 4 & -12 \\ 2 & 2 & -5 \end{bmatrix} \xrightarrow{R3(1, -1/2)} \begin{bmatrix} 1 & 0 & -1 \\ 6 & 4 & -12 \\ 2 & 2 & -5 \end{bmatrix}$$

$$\xrightarrow{R1(2, 1, -6), R1(3, 1, -2)} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & -6 \\ 0 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Set free variable $y_3 = 2 \Rightarrow y_2 = 3, y_1 = 2 \Rightarrow$ eigenvector $\mathbf{v}_1 = [2, 3, 2]^T$.

2. $\lambda_2 = -2$:

$$A + 2I = \begin{bmatrix} -1 & 0 & 2 \\ 6 & 5 & -12 \\ 2 & 2 & -4 \end{bmatrix} \xrightarrow{R1(2, 1, 6), R1(3, 1, 2)} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Set $y_3 = 1 \Rightarrow y_2 = 0, y_1 = 2 \Rightarrow$ eigenvector $\mathbf{v}_2 = [2, 0, 1]^T$



Exercise 9.1.27 (cont.)

3. $\lambda_3 = -3$:

$$A + 3I = \begin{bmatrix} 0 & 0 & 2 \\ 6 & 6 & -12 \\ 2 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Set free variable $y_2 = 1 \Rightarrow y_3 = 0$, $y_1 = -1 \Rightarrow$ eigenvector $\mathbf{v}_3 = [-1, 1, 0]^T$.

\Rightarrow **fundamental solution set:**

$$\mathbf{y}_1(t) = e^{-t} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \mathbf{y}_2(t) = e^{-2t} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{y}_3(t) = e^{-3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Note: Associated fundamental matrix is $Y(t) = \begin{bmatrix} 2e^{-t} & 2e^{-2t} & -e^{-3t} \\ 3e^{-t} & 0 & e^{-3t} \\ 2e^{-t} & e^{-2t} & 0 \end{bmatrix}$

General solution: $\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + c_3\mathbf{y}_3(t) = Y(t)\mathbf{c}$; $\mathbf{c} = [c_1, c_2, c_3]^T$



Exercise 9.1.36

Ex. 9.1.36: Find eigenvalues and eigenvectors using a computer for

$$A = \begin{bmatrix} -6 & 5 & -9 & 10 \\ 10 & -7 & 13 & -16 \\ 4 & -4 & 8 & -8 \\ -5 & 3 & -5 & 7 \end{bmatrix}$$

1. Numerical computation via Matlab's *poly*, *roots*, and *null* commands:

```
>> A=[-6 5 -9 10;10 -7 13 -16;4 -4 8 -8;-5 3 -5 7];cpol=poly(A)
cpol =
    1.0000    -2.0000    -1.0000     2.0000    -0.0000
```

The output of *poly* is a row vector whose entries are approximated values for the coefficients of the characteristic polynomial:

$$p(\lambda) \approx 1.0000 \times \lambda^4 - 2.0000 \times \lambda^3 - 1.0000 \times \lambda^2 + 2.0000 \times \lambda - 0.0000$$



Exercise 9.1.36 (cont.)

Find the roots of the characteristic polynomial:

```
>> evals=roots(cpol)
evals =
   -1.0000
    2.0000
    1.0000
    0.0000
```

So the eigenvalues (roots of $p(\lambda)$) are approximately -1.0000 , 2.0000 , 1.0000 , 0.0000 . They can be accessed via $evals(1)$, $evals(2)$ etc.

Now compute bases for the nullspaces of the eigenvalues using the *null*-command:

```
>> v1=null(A-evals(1)*eye(4))
v1 =
   -0.5774
    0.5774
    0.0000
   -0.5774
```

(The $n \times n$ identity matrix is denoted in Matlab by $eye(n)$ – here $n = 4$.) Analogously one can compute the other three eigenvectors.



Exercise 9.1.36 (cont.)

2. Symbolic computation using Matlab's *poly*, *factor* or *solve*, and *null* commands:

poly and *null* work also for symbolically defined matrices. The *roots* command works only for numerically defined vectors. To find roots of a symbolically defined polynomial, use the commands *factor* or *solve*.

```
>> sym_A=sym(A);sym_cpol=poly(sym_A)
sym_cpol =
x^4-2*x^3-x^2+2*x
```

Note that here the output is a symbolic polynomial expression with (default) variable x .



Exercise 9.1.36 (cont.)

You can find the eigenvalues with the *factor* command:

```
>> factor(sym_cpol)
ans =
x*(x-1)*(x-2)*(x+1)
```

So the exact eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = -1$.
Alternatively you can find them using *solve*:

```
>> sym_evals=solve(sym_cpol)
sym_evals =
[ 0]
[ 1]
[ 2]
[-1]
```



Exercise 9.1.36 (cont.)

Now find eigenvectors:

```
>> sym_v1=null(sym_A-sym_evals(1)*eye(4))
```

```
sym_v1 =
```

```
[ 1]
```

```
[ 1]
```

```
[ 1]
```

```
[ 1]
```

hence $\mathbf{v}_1 = [1, 1, 1, 1]^T$. Analogously one finds the eigenvectors for $\lambda_2, \lambda_3, \lambda_4$:

$$\mathbf{v}_2 = [0, -2, 0, 1]^T, \mathbf{v}_3 = [-1, 0, 2, 1]^T, \mathbf{v}_4 = [1, -1, 0, 1]^T.$$



Exercise 9.1.29

Ex. 9.1.29: Find eigenvalues and eigenvectors using a computer for

$$A = \begin{bmatrix} -7 & 2 & 10 \\ 0 & 1 & 0 \\ -5 & 2 & 8 \end{bmatrix}.$$

Eigenvalues and eigenvectors can be computed directly in Matlab with the *eig* command. Outputs:

V: matrix whose columns are eigenvectors

D: diagonal matrix whose diagonal entries are eigenvalues



Exercise 9.1.29 (cont.)

Without specification, outputs are floating point numbers:

```
A=[-7 2 10;0 1 0;-5 2 8];
[V,D]=eig(A)
V =
    -0.8944    -0.7071    -0.5774
         0         0         0.5774
    -0.4472    -0.7071    -0.5774
```

```
D =
    -2     0     0
     0     3     0
     0     0     1
```

Symbolic computation yields exact values if available:

```
A=[-7 2 10;0 1 0;-5 2 8];
[V,D]=eig(sym(A))
V =
 [ 1, 2, 1]
 [-1, 0, 0]
 [ 1, 1, 1]
```

```
D =
 [ 1, 0, 0]
 [ 0, -2, 0]
 [ 0, 0, 3]
```

$$\text{Hence } \lambda_1 = 1, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \lambda_2 = -2, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \lambda_3 = 3, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$



Exercise 9.1.39

Ex. 9.1.39: Find fundamental solution set via computer for $y' = Ay$ if

$$A = \begin{bmatrix} 20 & -34 & -10 \\ 12 & -21 & -5 \\ -2 & 4 & -2 \end{bmatrix}$$

Editing A in Matlab and applying Matlab's `eig` command to `sym(A)` yields the following eigenvalues and eigenvectors:

$\lambda_1 = -4$, $v_1 = [-1, -1, 1]^T$, $\lambda_2 = -2$, $v_2 = [2, 1, 1]^T$, $\lambda_3 = 3$, $v_3 = [2, 1, 0]^T$
 \Rightarrow fundamental solution set:

$$y_1(t) = e^{-4t}[-1, -1, 1]^T, \quad y_2(t) = e^{-2t}[2, 1, 1]^T, \quad y_3(t) = e^{3t}[2, 1, 0]^T$$



Exercise 9.1.49(a)

Ex. 9.1.49(i): Find determinant and eigenvalues of $A = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix}$ via computer

Describe any relationship between eigenvalues and determinant.

No computer necessary to find $\det(A) = -4$.

Eigenvalues (using Matlab): $\lambda_1 = 2$, $\lambda_2 = -2$, hence $\lambda_1\lambda_2 = -4 = \det(A)$.



Exercise 9.1.51(a)

Ex. 9.1.51(i): Find eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}$ via computer.

Describe any relationship between eigenvalues and triangular structure of A .
Matlab \rightarrow eigenvalues $\lambda_1 = 2$, $\lambda_2 = -4$. These are the diagonal entries of A .

Thm.: The eigenvalues of a lower or upper triangular matrix are the diagonal entries.

