

Math 3331 Differential Equations

9.3 Phase Plane Portraits

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9.3 Phase Plane Portraits

- Classification of 2d Systems
- Distinct Real Eigenvalues
 - Phase Portrait
 - Saddle: $\lambda_1 > 0 > \lambda_2$
 - Nodal Source: $\lambda_1 > \lambda_2 > 0$
 - Nodal Sink: $\lambda_1 < \lambda_2 < 0$
- Complex Eigenvalues
 - Center: $\alpha = 0$
 - Spiral Source: $\alpha > 0$
 - Spiral Sink: $\alpha < 0$
- Borderline Cases
 - Degenerate Node: Borderline Case Spiral/Node
 - Degenerate Nodal Source
 - Degenerate Nodal Sink
 - SaddleNode: Borderline Case Node/Saddle
 - Unstable Saddle-Node
 - Stable Saddle-Node



Classification of 2d Systems

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$T = a + d, \quad D = ad - bc$$

$$p(\lambda) = \lambda^2 - T\lambda + D$$

Case A: $T^2 - 4D > 0$

\Rightarrow real distinct eigenvalues

$$\lambda_{1,2} = (T \pm \sqrt{T^2 - 4D})/2$$

Case B: $T^2 - 4D < 0$

\Rightarrow complex eigenvalues

$$\lambda_{1,2} = \alpha \pm i\beta$$

$$\alpha = T/2, \quad \beta = \sqrt{4D - T^2}/2$$

Case C: $T^2 - 4D = 0$

\Rightarrow single eigenvalue

$$\lambda = T/2$$



Case A: $T^2 - 4D > 0$

Case A: $T^2 - 4D > 0$

\Rightarrow real distinct eigenvalues

$$\lambda_{1,2} = (T \pm \sqrt{T^2 - 4D})/2$$

General Solution:

($\mathbf{v}_1, \mathbf{v}_2$: eigenvectors)

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2$$

$L_{1,2}$: Full lines generated by $\mathbf{v}_{1,2}$

Half line trajectories:

if $c_2 = 0 \Rightarrow \mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1$

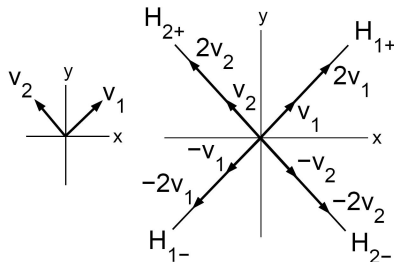
\Rightarrow trajectory is half line

$$H_{1+} = \{\mathbf{x} = \alpha \mathbf{v}_1 \mid \alpha > 0\} \text{ if } c_1 > 0$$

$$H_{1-} = \{\mathbf{x} = \alpha \mathbf{v}_1 \mid \alpha < 0\} \text{ if } c_1 < 0$$

Same for $H_{2\pm}$ if $c_1 = 0$, $c_2 > 0$ or < 0

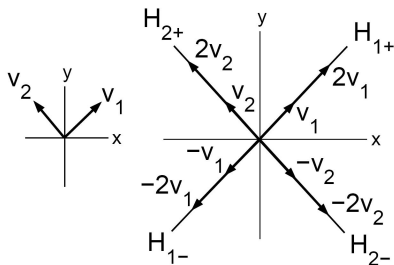
- The 4 half line trajectories separate 4 regions of \mathbb{R}^2



Phase Portrait

Phase portrait:

Sketch trajectories. Indicate *direction of motion* by arrows pointing in the direction of increasing t



Direction of Motion on Half Line Trajectories:

- If $\lambda_1 > 0$ then $x(t) = c_1 e^{\lambda_1 t} v_1$
 - moves out to ∞ for $t \rightarrow \infty$ (outwards arrow on H_{1+})
 - approaches 0 for $t \rightarrow -\infty$
- If $\lambda_1 < 0$ then $x(t) = c_1 e^{\lambda_1 t} v_1$
 - approaches 0 for $t \rightarrow \infty$ (inwards arrow on H_{1+})
 - moves out to ∞ for $t \rightarrow -\infty$

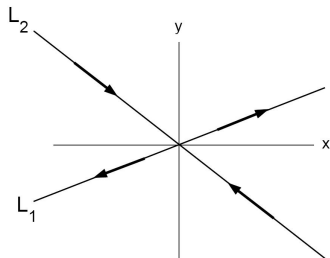


Saddle: $\lambda_1 > 0 > \lambda_2$

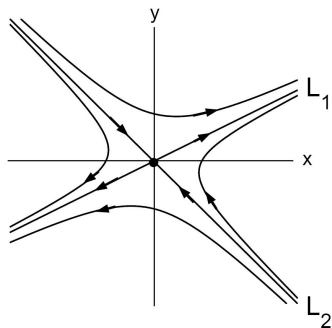
Saddle

$$\lambda_1 > 0 > \lambda_2$$

Half line trajectories



Generic Trajectories



Generic trajectory in each region approaches

- L_1 for $t \rightarrow \infty$
- L_2 for $t \rightarrow -\infty$

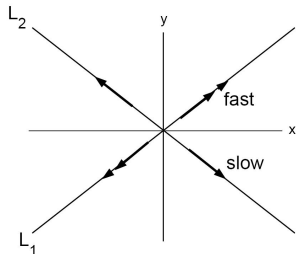


Nodal Source: $\lambda_1 > \lambda_2 > 0$

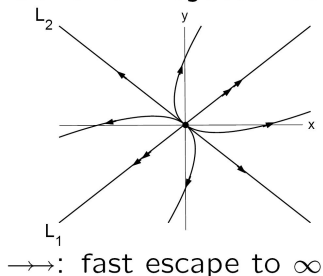
Nodal source

$$\lambda_1 > \lambda_2 > 0$$

Half line trajectories



Generic Trajectories



Generic trajectory is

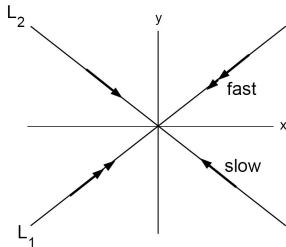
- parallel to L_1 for $t \rightarrow \infty$
- tangent to L_2 for $t \rightarrow -\infty$

Nodal Sink: $\lambda_1 < \lambda_2 < 0$

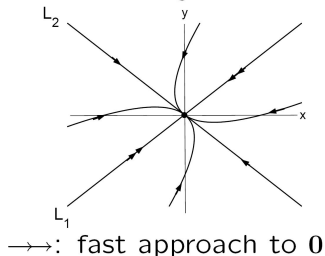
Nodal sink

$$\lambda_1 < \lambda_2 < 0$$

Half line trajectories



Generic Trajectories



Generic trajectory is

- parallel to L_1 for $t \rightarrow -\infty$
- tangent to L_2 for $t \rightarrow \infty$

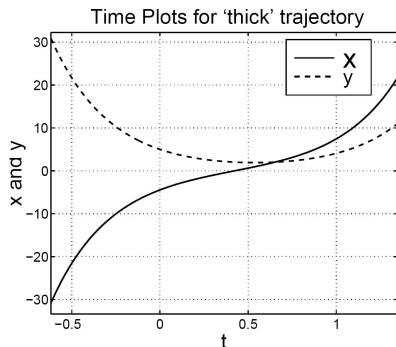
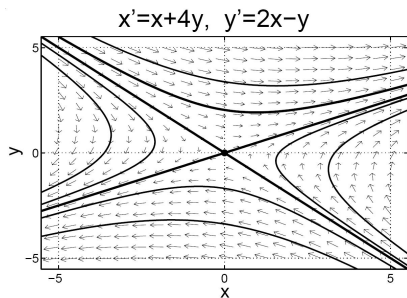
Saddle: Example

Saddle

$$\text{Ex.: } A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$$

$$\lambda_1 = 3 \leftrightarrow \mathbf{v}_1 = [2, 1]^T$$

$$\lambda_2 = -3 \leftrightarrow \mathbf{v}_2 = [-1, 1]^T$$



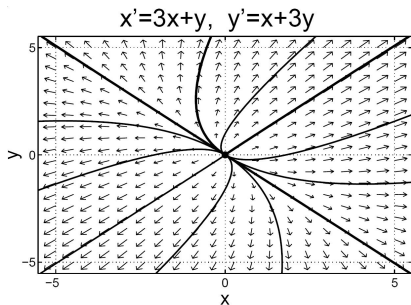
Nodal Source: Example

Nodal Source

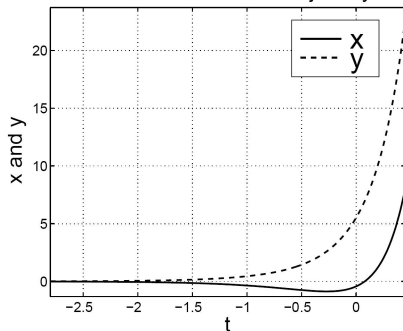
$$\text{Ex.}: A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\lambda_1 = 4 \leftrightarrow \mathbf{v}_1 = [1, 1]^T$$

$$\lambda_2 = 2 \leftrightarrow \mathbf{v}_2 = [-1, 1]^T$$



Time Plots for 'thick' trajectory



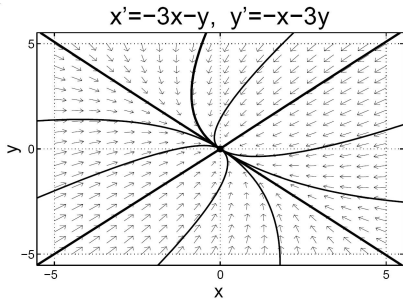
Nodal Sink: Example

Nodal Sink

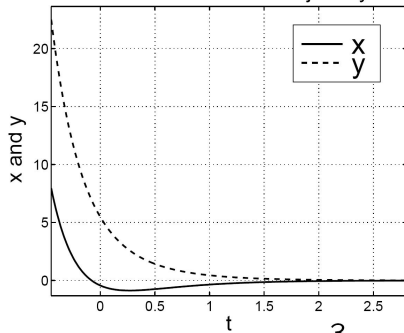
$$\text{Ex.: } A = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}$$

$$\lambda_1 = -4 \leftrightarrow \mathbf{v}_1 = [1, 1]^T$$

$$\lambda_2 = -2 \leftrightarrow \mathbf{v}_2 = [-1, 1]^T$$



Time Plots for 'thick' trajectory



Case A: $T^2 - 4D < 0$

Case B: $T^2 - 4D < 0$

\Rightarrow complex eigenvalues

$$\lambda_{1,2} = \alpha \pm i\beta$$

$$\alpha = T/2, \beta = \sqrt{4D - T^2}/2$$

λ complex

\Rightarrow eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ complex

\Rightarrow no half line solutions

General solution:

$$\mathbf{x}(t) = e^{at} \left[c_1(\mathbf{u} \cos \beta t - \mathbf{w} \sin \beta t) + c_2(\mathbf{u} \sin \beta t + \mathbf{w} \cos \beta t) \right]$$

Subcases of Case B

Center: $\alpha = 0$

Spiral Source: $\alpha > 0$

Spiral Sink: $\alpha < 0$

Borderline Case:

Center ($\alpha = 0$) is border between spiral source ($\alpha > 0$) and spiral sink ($\alpha < 0$).

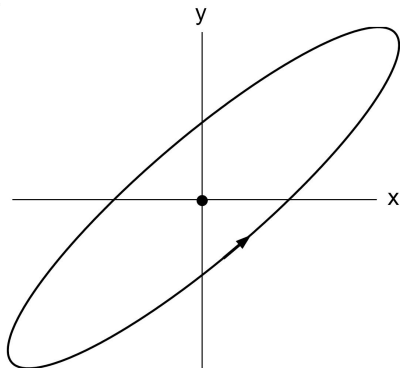


Center: $\alpha = 0$

Center: $\alpha = 0$

$\Rightarrow \mathbf{x}(t)$ periodic

\Rightarrow trajectories are closed curves



Direction of Rotation: At $\mathbf{x} = [1, 0]^T$, $y' = c$.

If $c > 0$, \Rightarrow counterclockwise,

If $c < 0$, \Rightarrow clockwise.

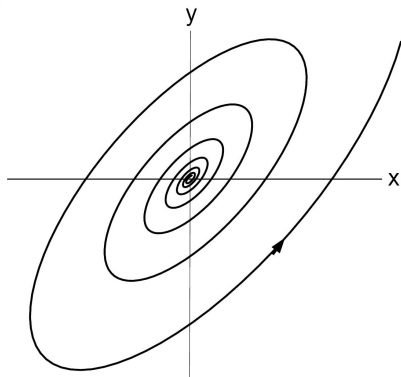


Spiral Source: $\alpha > 0$

Spiral Source: $\alpha > 0$

\Rightarrow growing oscillations

\Rightarrow trajectories are
outgoing spirals



Direction of Rotation: At $\mathbf{x} = [1, 0]^T$, $y' = c$.

If $c > 0$, \Rightarrow counterclockwise,

If $c < 0$, \Rightarrow clockwise.

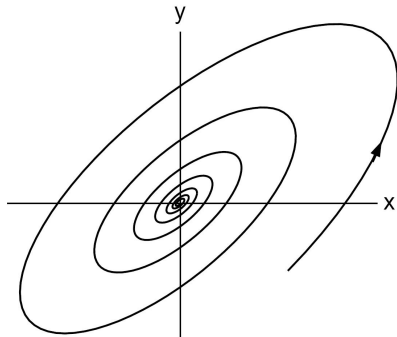


Spiral Sink: $\alpha < 0$

Spiral Sink: $\alpha < 0$

\Rightarrow decaying oscillations

\Rightarrow trajectories are
ingoing spirals



Direction of Rotation: At $\mathbf{x} = [1, 0]^T$, $y' = c$.

If $c > 0$, \Rightarrow counterclockwise,

If $c < 0$, \Rightarrow clockwise.

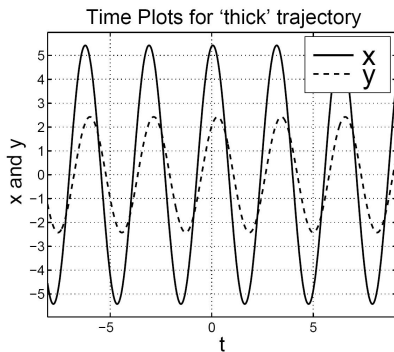
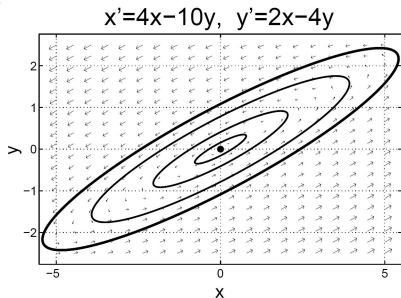


Center: Example

Center

$$\text{Ex.: } A = \begin{bmatrix} 4 & -10 \\ 2 & -4 \end{bmatrix}$$

$$\lambda = 2i \leftrightarrow \mathbf{v} = \begin{bmatrix} 2 + i \\ 1 \end{bmatrix}$$

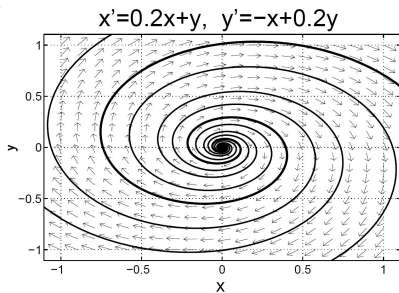


Spiral Source: Example

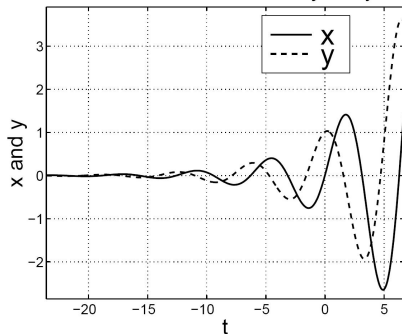
Spiral Source

$$\text{Ex.: } A = \begin{bmatrix} 0.2 & 1 \\ -1 & 0.2 \end{bmatrix}$$

$$\lambda = 0.2 + i \leftrightarrow \mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$



Time Plots for 'thick' trajectory

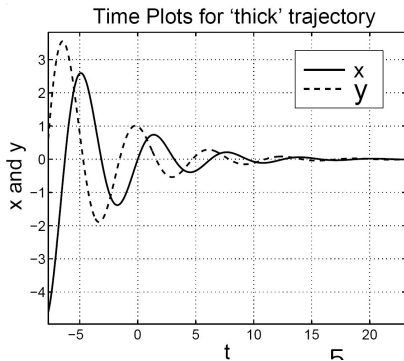
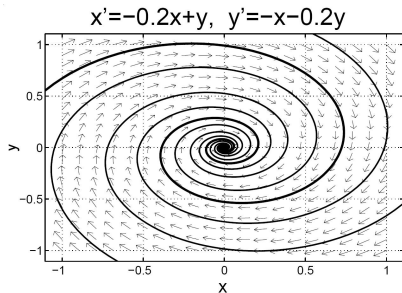


Spiral Sink: Example

Spiral Sink

$$\text{Ex.: } A = \begin{bmatrix} -0.2 & 1 \\ -1 & -0.2 \end{bmatrix}$$

$$\lambda = -0.2 + i \leftrightarrow \mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$



Degenerate Node: Borderline Case Spiral/Node

Degenerate Node: Borderline Case Spiral/Node

- Assume $T^2 - 4D = 0 \Rightarrow$ single eigenvalue $\lambda = T/2$
- Assume generic case: $(A - \lambda I) \neq 0 \Rightarrow$ single eigenvector \mathbf{v}
- Let $(A - \lambda I)\mathbf{w} = \mathbf{v} \Rightarrow$ General solution:

$$\mathbf{x}(t) = c_1 e^{\lambda t} \mathbf{v} + c_2 e^{\lambda t} (\mathbf{w} + t\mathbf{v})$$

\Rightarrow only two half line solutions on straight line generated by \mathbf{v}

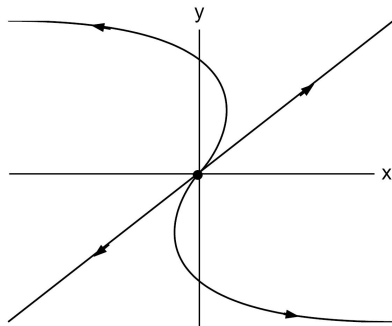


Degenerate Nodal Source

Degenerate Nodal Source:

$$T > 0$$

borderline case $\left\{ \begin{array}{l} \text{nodal source} \\ \text{spiral source} \end{array} \right.$

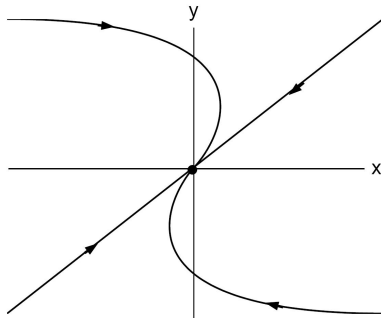


Degenerate Nodal Sink

Degenerate Nodal Sink:

$$T < 0$$

borderline case $\left\{ \begin{array}{l} \text{nodal sink} \\ \text{spiral sink} \end{array} \right.$



SaddleNode: Borderline Case Node/Saddle

SaddleNode: Borderline Case Node/Saddle

- Assume $D = 0$, $T \neq 0 \Rightarrow$ eigenvalues $\lambda_1 = 0$, $\lambda_2 = T$
- Let $\mathbf{v}_1, \mathbf{v}_2$ be the eigenvectors \Rightarrow General solution:

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2$$

- \Rightarrow
- line of equilibrium points generated by \mathbf{v}_1
 - infinitely many half line solutions on straight lines parallel to line generated by \mathbf{v}_2

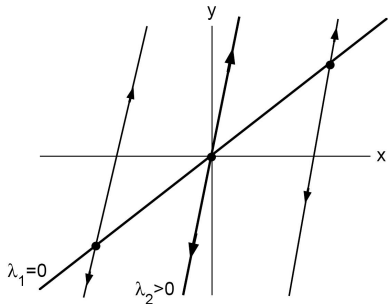


Unstable Saddle-Node

Unstable Saddle-Node:

$$T > 0$$

borderline case $\left\{ \begin{array}{l} \text{nodal source} \\ \text{saddle} \end{array} \right.$



Stable Saddle-Node

Stable Saddle-Node:

$$T < 0$$

borderline case $\left\{ \begin{array}{l} \text{nodal sink} \\ \text{saddle} \end{array} \right.$

