Math 3331 Differential Equations

9.5 Higher-Dimensional Systems

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9.5 Higher-Dimensional Systems

- Homogenous System: Distinct Real Eigenvalues
- Homogenous System: Complex Eigenvalues
- Fundamental Set of Eigenvector Solutions
- Examples

Homogenous System: Distinct Real Eigenvalues

Homogenous system:

$$\mathbf{x}' = A\mathbf{x}, \quad A : n \times n$$

Characteristic Polynomial:
(degree n)

$$p(\lambda) = \det(A - \lambda I)$$

Fundamental Thm. of Algebra:
If the roots are counted with multiplicities, then $p(\lambda)$ has exactly n roots $\lambda_1, \ldots, \lambda_n$, and

$$p(\lambda) = (-1)^n(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$$

Thm.: If $\lambda_1, \ldots, \lambda_n$ are n real eigenvalues of A and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n linearly independent eigenvectors, then

$$\mathbf{x}_j(t) = e^{\lambda_j t}\mathbf{v}_j, \; 1 \le j \le n$$

are a fundamental set of solutions.

Homogenous System: Complex Eigenvalues

A: real $n \times n$-matrix

System: $\mathbf{x}' = A\mathbf{x}$ (1)

Assume $A\mathbf{v} = \lambda\mathbf{v}$ with

$$\begin{aligned} \lambda &= \alpha + i\beta \in \mathbf{C}, \quad \beta \neq 0 \\ \mathbf{v} &= \mathbf{u} + i\mathbf{w} \in \mathbf{C}^n, \quad \mathbf{v} \neq \mathbf{0} \end{aligned}$$

Pairs: $A\mathbf{v} = \lambda\mathbf{v} \Rightarrow A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$

$\Rightarrow$ complex conjugate pairs of eigenvalues and eigenvectors

Thm.: Let λ be a complex eigenvalue with eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$. Then $\mathbf{v}, \overline{\mathbf{v}}$ and $\mathbf{u}, \mathbf{w}$ are linearly independent.

Linearly independent
complex solutions of (1):

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{v}, \quad \overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{v}}$$

Real and imaginary parts:

$$\begin{aligned} \mathbf{z}(t) &= e^{(\alpha+i\beta)t}(\mathbf{u} + i\mathbf{w}) \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u} + i\mathbf{w}) \\ &= e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{w}\sin\beta t) \\ &\quad + ie^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{w}\cos\beta t) \\ &= \operatorname{Re}\mathbf{z}(t) + i\operatorname{Im}\mathbf{z}(t) \end{aligned}$$

Linearly independent
real solutions of (1):

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{w}\sin\beta t) \\ \mathbf{x}_2(t) &= e^{\alpha t}(\mathbf{u}\sin\beta t + \mathbf{w}\cos\beta t) \end{aligned}$$

Fundamental Set of Eigenvector Solutions

A: real $n \times n$-matrix

System: $\mathbf{x}' = A\mathbf{x}$ (1)

Thm.: Assume

1. A has k real eigenvalues $\lambda_1, \ldots, \lambda_k$ with linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$.
2. A has m complex conjugate pairs $\lambda_{k+1}, \overline{\lambda}_{k+1}, \ldots, \lambda_{k+m}, \overline{\lambda}_{k+m}$ of eigenvalues with eigenvectors $\mathbf{v}_{k+1}, \overline{\mathbf{v}}_{k+1}, \ldots, \mathbf{v}_{k+m}, \overline{\mathbf{v}}_{k+m}$.
3. $n = k + 2m$ and the eigenvectors of 2. are linearly independent.

Then the n vector functions

$$\begin{aligned} \mathbf{x}_i(t) &= e^{\lambda_i t}\mathbf{v}_i, \quad 1 \le i \le k \\ \mathbf{x}_j(t) &= e^{\alpha_j t}(\mathbf{u}_j \cos\beta_j t - \mathbf{w}_j \sin\beta_j t) \\ \mathbf{x}_{j+m}(t) &= e^{\alpha_j t}(\mathbf{u}_j \sin\beta_j t + \mathbf{w}_j \cos\beta_j t) \\ & \quad k+1 \le j \le k+m \end{aligned}$$

are a fundamental set of solutions.

Example

Ex.: $A = \begin{bmatrix} 14 & 8 & -19 \\ -40 & -25 & 52 \\ -5 & -4 & 6 \end{bmatrix}$

Use Matlab's *poly* and *factor*

$\Rightarrow p(\lambda) = -(\lambda+1)[(\lambda+2)^2+9]$

$\Rightarrow$ eigenvalues: $\lambda_1 = -1$
$\lambda_2 = -2+3i$, $\lambda_3 = \overline{\lambda_2}$

eigenvectors (using Matlab's *null*):

$$\begin{aligned} \mathbf{v}_1 &= [2,1,2]^T \\ \mathbf{v}_2 &= [i, 2-2i, 1]^T \\ &= [0,2,1]^T + i[1,-2,0]^T \end{aligned}$$

Fundamental set of solutions:

$$\begin{aligned} \mathbf{x}_1(t) &= e^{-t}[2,1,2]^T \\ \mathbf{x}_2(t) &= e^{-2t}([0,2,1]^T \cos 3t \\ &\quad -[1,-2,0]^T \sin 3t) = e^{-2t}\mathbf{p}(t) \\ \mathbf{x}_3(t) &= e^{-2t}([0,2,1]^T \sin 3t \\ &\quad +[1,-2,0]^T \cos 3t) = e^{-2t}\mathbf{q}(t) \end{aligned}$$

$\mathbf{p}(t) = [-\sin 3t, 2\cos 3t + 2\sin 3t, \cos 3t]^T$

$\mathbf{q}(t) = [\cos 3t, 2\sin 3t - 2\cos 3t, \sin 3t]^T$