

Math 3331 Differential Equations

9.6 The Exponential of a Matrix

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9.6 The Exponential of a Matrix

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Fundamental Matrix

$$\mathbf{x}' = A\mathbf{x}, \quad A : n \times n \quad (1)$$

Def.: If $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ is a fundamental set of solutions (F.S.S.) of (1), then

$X(t) = [\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)]$ ($n \times n$) is called a fundamental matrix (F.M.) for (1).

General solution:

$$(\mathbf{c} = [c_1, \dots, c_n]^T)$$

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}_1(t) + \dots + c_n\mathbf{x}_n(t) \\ &= X(t)\mathbf{c} \end{aligned}$$

Thm.: If $X(t)$ is a F.M. for (1) and C is a constant nonsingular matrix, then $X(t)C$ is also a F.M.

Proof: Each column of $X(t)C$ is a linear combination of the columns of $X(t)$ and so is a solution of (1), and $X(0)C$ is nonsingular.



Example

$$\text{Ex.: } A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$$

Eigenvalues and eigenvectors:

$$\lambda_1 = -1 \leftrightarrow \mathbf{v}_1 = [2, 3]^T$$

$$\lambda_2 = -2 \leftrightarrow \mathbf{v}_2 = [1, 1]^T$$

F.S.S.:

$$\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{x}_2(t) = e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{F.M.: } X(t) = \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix}$$

If we set

$$y_1(t) = 2x_2(t), \quad y_2(t) = 3x_2(t),$$

$y_1(t), y_2(t)$ are also F.S.S. with F.M.

$$\begin{aligned} Y(t) &= \begin{bmatrix} 3e^{-2t} & 4e^{-t} \\ 3e^{-2t} & 6e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \end{aligned}$$



Matrix Exponential

Consider IVP:

$$\mathbf{x}' = A\mathbf{x}, \mathbf{x}(0) = \mathbf{x}_0 \quad (2)$$

Solution of IVP: If $X(t)$ is a F.M., the general solution is

$$\mathbf{x}(t) = X(t)\mathbf{c}$$

Match \mathbf{c} to IC:

$$\begin{aligned} \mathbf{x}(0) &= X(0)\mathbf{c} = \mathbf{x}_0 \\ \Rightarrow \mathbf{c} &= (X(0))^{-1}\mathbf{x}_0 \\ \Rightarrow \mathbf{x}(t) &= X(t)(X(0))^{-1}\mathbf{x}_0 \end{aligned}$$

Def.: Given a F.M. $X(t)$, then

$$e^{At} \stackrel{\text{def}}{=} X(t)(X(0))^{-1}$$

is the matrix exponential of At .

Thm.: The solution of (2) is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$



Example: Matrix Exponential

$$\mathbf{Ex.}: A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$$

$$\text{F.M.: } X(t) = \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}, (X(0))^{-1} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$$

$$\begin{aligned} e^{At} &= X(t)(X(0))^{-1} \\ &= \begin{bmatrix} 2e^{-t} & e^{-2t} \\ 3e^{-t} & e^{-2t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 3e^{-2t} - 2e^{-t} & 2e^{-t} - 2e^{-2t} \\ 3e^{-2t} - 3e^{-t} & 3e^{-t} - 2e^{-2t} \end{bmatrix} \end{aligned}$$



Example: IVP

$$\mathbf{IVP:} \quad \mathbf{x}' = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solution:

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} 3e^{-2t} - 2e^{-t} & 2e^{-t} - 2e^{-2t} \\ 3e^{-2t} - 3e^{-t} & 3e^{-t} - 2e^{-2t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4e^{-2t} - 2e^{-t} \\ 4e^{-2t} - 3e^{-t} \end{bmatrix} \end{aligned}$$



Properties of the Matrix Exponential (I)

- Exponential series ($A^0 = I$):

$$e^{At} = \sum_{m=0}^{\infty} (At)^m / m!$$

Convergence for any matrix A

- $\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$

- If $D = [d_{ij}]$ is a diagonal matrix ($d_{ij} = 0$ for $i \neq j$), then e^{Dt} is a diagonal matrix with entries $e^{d_{ii}t}$. Ex.:

$$\exp\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} t\right) = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$$

- Special case ($d_{ii} = r$):

$$e^{(rI)t} = e^{rt} I$$



Properties of the Matrix Exponential (II)

- If $AB = BA$, then

$$e^{(A+B)t} = e^{At}e^{Bt} = e^{Bt}e^{At}$$

Note: If $AB \neq BA$, then in general

$$e^{(A+B)t} \neq e^{At}e^{Bt} \neq e^{Bt}e^{At}$$

- e^{At} is nonsingular, and

$$(e^{At})^{-1} = e^{-At}$$

- If V is nonsingular, then

$$e^{(VAV^{-1})t} = Ve^{At}V^{-1}$$

- If \mathbf{v} is an eigenvector for an eigenvalue λ , then

$$e^{At}\mathbf{v} = e^{\lambda t}\mathbf{v}$$



Matrices With Only One Eigenvalue

Thm.: If A has only one eigenvalue λ , then there is an integer k , $0 < k \leq n$, such that

$$(A - \lambda I)^k = 0$$

Use this to compute e^{At} as follows. Write $A = \lambda I + (A - \lambda I)$. Then

$$\begin{aligned} e^{At} &= e^{(\lambda I)t + (A - \lambda I)t} \\ &= e^{(\lambda I)t} e^{(A - \lambda I)t} \\ &= e^{\lambda t} e^{(A - \lambda I)t} \\ &= e^{\lambda t} \sum_{j=0}^{k-1} (A - \lambda I)^j (t^j / j!) \end{aligned}$$

\Rightarrow only k terms of exponential series required



Example

Ex.: $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$; $T = -2$, $D = 1$

$$\begin{aligned} p(\lambda) &= \lambda^2 - T\lambda + D \\ &= \lambda^2 + 2\lambda + 1 \\ &= (\lambda + 1)^2 \end{aligned}$$

\Rightarrow only one eigenvalue $\lambda = -1$

$$\begin{aligned} A + I &= \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \\ (A + I)^2 &= \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (k = 2) \end{aligned}$$

$$\begin{aligned} \Rightarrow e^{(A+I)t} &= I + (A + I)t \\ &= \begin{bmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \Rightarrow e^{At} &= e^{-t} e^{(A+I)t} \\ &= e^{-t} \begin{bmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{bmatrix} \end{aligned}$$



Generalized Eigenvectors

If A has repeated eigenvalues, n linearly independent eigenvectors may not exist \Rightarrow need generalized eigenvectors.



Generalized Eigenvectors: Definition

Def.: Let λ be eigenvalue of A .

(a) The algebraic multiplicity, m , of λ is the multiplicity of λ as root of the characteristic polynomial (CN Sec. 9.5).

(b) The geometric multiplicity, m_g , of λ is $\dim \text{null}(A - \lambda I)$.

Need: m linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$ associated with λ .

- If $m_g = m \Rightarrow m$ linearly independent eigenvector solutions.
- What if $m_g < m$?

Thm.: If λ is an eigenvalue with algebraic multiplicity m , then there is an integer k , $0 < k \leq m$, such that

$$\begin{aligned} \dim \text{null}((A - \lambda I)^k) &= m \\ \dim \text{null}((A - \lambda I)^{k-1}) &< m \end{aligned}$$

Def.: Any nonzero vector \mathbf{v} in $\text{null}((A - \lambda I)^k)$ is a generalized eigenvector for λ .

Solution associated with \mathbf{v} :

$$\begin{aligned} (A - \lambda I)^k \mathbf{v} &= \mathbf{0} \Rightarrow \\ e^{At} \mathbf{v} &= e^{\lambda t} \sum_{j=0}^{k-1} (t^j / j!) (A - \lambda I)^j \mathbf{v} \end{aligned}$$



Generalized Eigenvectors and Associated Solutions

Thm.: Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be a basis of $\text{null}(A - \lambda I)^k$. Then the

$$\mathbf{x}_i(t) = e^{\lambda t} \sum_{j=0}^{k-1} (t^j / j!) (A - \lambda I)^j \mathbf{v}_i,$$

$1 \leq i \leq m$, are m linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$.



Example: 2d Systems

2d Systems: (Sec. 9.2)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left\{ \begin{array}{l} T = a + d \\ D = ad - bc \end{array} \right\}$$

Assume $T^2 - 4D = 0 \Rightarrow$

$$p(\lambda) = (\lambda - \lambda_1)^2, \quad \lambda_1 = T/2$$

(a) If $A = \lambda_1 I \Rightarrow m_g = 2$

$$\Rightarrow \mathbf{x}(t) = e^{\lambda_1 t} \mathbf{x}(0)$$

(any vector is eigenvector)

(b) If $A \neq \lambda_1 I \Rightarrow m_g = 1:$

- Compute eigenvector \mathbf{v}
- Pick vector \mathbf{w} that is *not* a multiple of \mathbf{v}
 $\Rightarrow (A - \lambda_1 I)\mathbf{w} = a\mathbf{v}$
 for some $a \neq 0$ (any $\mathbf{w} \in \mathbf{R}^2$
 is generalized eigenvector)

• \Rightarrow F.S.S.:

$$\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}$$

$$\mathbf{x}_2(t) = e^{\lambda_1 t} (\mathbf{w} + a\mathbf{v}t)$$



Example 1

Ex.: $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$: $T = -2$, $D = 1$
 $\Rightarrow T^2 - 4D = 0 \Rightarrow$ eigenvalue $\lambda = -1$

$$A + I = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

\Rightarrow eigenvector $\mathbf{v} = [-2, 1]^T$

Choose $\mathbf{w} = [1, 0]^T$ (simple form) \Rightarrow

$$(A+I)\mathbf{w} = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -\mathbf{v}$$

\Rightarrow F.S.S.:

$$\begin{aligned} \mathbf{x}_1(t) &= e^{-t}\mathbf{v} = e^{-t}[-2, 1]^T \\ \mathbf{x}_2(t) &= e^{-t}(\mathbf{w} - t\mathbf{v}) \\ &= e^{-t}([1, 0]^T - t[-2, 1]^T) \\ &= e^{-t}[1 + 2t, -t]^T \end{aligned}$$

Other Method: Compute (c.f. p.4)

$$e^{At} = e^{-t}(I + (A+I)t) = e^{-t} \begin{bmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{bmatrix}$$

Columns of e^{At} are also F.S.S.



Example 2

$$\text{Ex.: } A = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -1 & 0 \\ -1 & -3 & -3 \end{bmatrix}$$

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} -1-\lambda & 2 & 1 \\ 0 & -1-\lambda & 0 \\ -1 & -3 & -3-\lambda \end{vmatrix} \\ &= (-1-\lambda) \begin{vmatrix} -1-\lambda & 1 \\ -1 & -3-\lambda \end{vmatrix} \\ &= (-1-\lambda)[(1+\lambda)(3+\lambda) + 1] \\ &= -(\lambda+1)(\lambda^2 + 4\lambda + 4) \\ &= -(\lambda+1)(\lambda+2)^2 \end{aligned}$$

$$\Rightarrow \text{eigenvalues } \lambda_1 = -1, m_1 = 1 \\ \lambda_2 = -2, m_2 = 2$$

Compute $A - \lambda_1 I = A + I$:

$$A+I = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -3 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Set $x_3 = -2$

$$\Rightarrow \text{eigenvector } \mathbf{v}_1 = [1, 1, -2]^T$$

Since $m_1 = 1$

\Rightarrow one (eigenvector) solution:

$$\mathbf{x}_1(t) = e^{-t}[1, 1, -2]^T$$



Example 2 (cont.)

$m_2 = 2 \rightarrow$ check $A - \lambda_2 I$, $(A - \lambda_2 I)^2$:

$$A + 2I = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A + 2I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A + 2I \rightarrow$ eigenvector $\mathbf{v}_2 = [1, 0, -1]^T$
 \Rightarrow eigenvector solution:

$$\mathbf{x}_2(t) = e^{-2t}[1, 0, -1]^T$$

For $\mathbf{x}_3(t)$ use generalized eigenvector \mathbf{v}_3 that is *not* an eigenvector.

Basis of $\text{null}((A + 2I)^2)$: $\begin{cases} \mathbf{u}_1 = [1, 0, 0]^T \\ \mathbf{u}_2 = [0, 0, 1]^T \end{cases}$

Note: $\mathbf{v}_2 = \mathbf{u}_1 - \mathbf{u}_2$

$$= (A + 2I)\mathbf{u}_1 = (A + 2I)\mathbf{u}_2$$

\mathbf{v}_3 can be any vector $c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ s.t.

$$(A + 2I)(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) = (c_1 + c_2)\mathbf{v}_2 \neq 0$$

Choose $\mathbf{v}_3 = \mathbf{u}_2 = [0, 0, 1]^T$ (text: \mathbf{u}_1)

$$\Rightarrow \mathbf{x}_3(t) = e^{-2t}(I\mathbf{v}_3 + t(A + 2I)\mathbf{v}_3)$$

$$= e^{-2t}(\mathbf{v}_3 + t\mathbf{v}_2)$$

$$= e^{-2t}[t, 0, 1 - t]^T \quad 7$$



Example 3

$$\text{Ex.: } A = \begin{bmatrix} 6 & 6 & -3 & 2 \\ -4 & -4 & 2 & 0 \\ 8 & 7 & -4 & 4 \\ 1 & 0 & -1 & -2 \end{bmatrix}$$

Matlab $\rightarrow p(\lambda) = ((\lambda + 1)^2 + 1)^2$

\Rightarrow single complex pair of eigenvalues

$$\lambda_1 = -1 + i, \lambda_2 = \overline{\lambda_1} \quad (m = 2).$$

1. Check $B \equiv A - \lambda_1 I = A - (-1 + i)I$:

$$B = \begin{bmatrix} 7 - i & 6 & -3 & 2 \\ -4 & -3 - i & 2 & 0 \\ 8 & 7 & -3 - i & 4 \\ 1 & 0 & -1 & -1 - i \end{bmatrix}$$

Matlab \rightarrow basis for $\text{null}(B)$:

$$\mathbf{v}_1 = [2, 0, 4, -1 + i]^T$$

\Rightarrow Complex eigenvector solution:

$$\mathbf{z}_1(t) = e^{(-1+i)t} [2, 0, 4, -1 + i]^T$$

2. Check $B^2 = (A - \lambda_1 I)^2$:

$$B^2 = \begin{bmatrix} 2 - 14i & 3 - 12i & -2 + 6i & -4i \\ 8i & -2 + 6i & -4i & 0 \\ 8 - 16i & 6 - 14i & -6 + 6i & -8i \\ -2 - 2i & -1 & 1 + 2i & -2 + 2i \end{bmatrix}$$

Matlab \rightarrow basis for $\text{null}(B^2)$:

$$\mathbf{u}_1 = [2, 0, 4, -1 + i]^T = \mathbf{v}_1$$

$$\mathbf{u}_2 = [-3 - i, 4, 0, -2 + 2i]^T$$

$\Rightarrow \mathbf{u}_2$ is generalized eigenvector that is not an eigenvector.

Pick $\mathbf{v}_3 = \mathbf{u}_2 = [-3 - i, 4, 0, -2 + 2i]^T$

Need: $B\mathbf{v}_3 = [-2, 0, -4, 1 - i]^T = -\mathbf{v}_2$

Complex solution associated with \mathbf{v}_3 :

$$\begin{aligned} \mathbf{z}_2(t) &= e^{(-1+i)t} (I\mathbf{v}_3 + tB\mathbf{v}_3) \\ &= e^{(-1+i)t} (\mathbf{v}_3 - t\mathbf{v}_2) \\ &= e^{(-1+i)t} [-3 - i - 2t, 4, -4t, \\ &\quad -2 + 2i + (1 - i)t]^T \end{aligned}$$



Example 3 (cont.)

3. Take real and imaginary parts of $\mathbf{z}_1(t)$ and $\mathbf{z}_2(t)$ to obtain F.S.S:

$$\mathbf{x}_1(t) = \operatorname{Re} \mathbf{z}_1(t) = e^{-t}[2 \cos t, 0, 4 \cos t, -\cos t - \sin t]^T$$

$$\mathbf{x}_2(t) = \operatorname{Im} \mathbf{z}_1(t) = e^{-t}[2 \sin t, 0, 4 \sin t, \cos t - \sin t]^T$$

$$\mathbf{x}_3(t) = \operatorname{Re} \mathbf{z}_2(t) = e^{-t}[\sin t - (3 + 2t) \cos t, 4 \cos t, -4t \cos t, (t - 2)(\cos t + \sin t)]^T$$

$$\mathbf{x}_4(t) = \operatorname{Im} \mathbf{z}_2(t) = e^{-t}[-\cos t - (3 + 2t) \sin t, -4 \sin t, -4t \sin t, (t - 2)(\sin t - \cos t)]^T$$



Example 4

Ex.: $A = \begin{bmatrix} 7 & 5 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 12 & 10 & -5 & 4 \\ -4 & -4 & 2 & -1 \end{bmatrix}$

Matlab $\rightarrow p(\lambda) = (\lambda + 1)(\lambda - 1)^3$

\Rightarrow eigenvalues $\lambda_1 = -1, m_1 = 1$
 $\lambda_2 = 1, m_2 = 3$

Find eigenvector for λ_1 :

$$A + I = \begin{bmatrix} 8 & 5 & -3 & 2 \\ 0 & 2 & 0 & 0 \\ 12 & 10 & -4 & 4 \\ -4 & -4 & 2 & 0 \end{bmatrix}$$

Matlab \rightarrow eigenvector (basis vector for $\text{null}(A + I)$): $\mathbf{v}_1 = [1, 0, 2, -1]^T$

Associated eigenvector solution:

$$\mathbf{x}_1(t) = e^{-t}[1, 0, 2, -1]^T$$

For $\lambda_2 = 1 \rightarrow$ check powers of $A - I$:

$$B \equiv A - I = \begin{bmatrix} 6 & 5 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 12 & 10 & -6 & 4 \\ -4 & -4 & 2 & -2 \end{bmatrix}$$

Matlab \rightarrow basis of $\text{null}(B)$:

$$\mathbf{v}_2 = [1, 0, 2, 0]^T$$

$$\mathbf{v}_3 = [1, -2, 0, 2]^T$$

Associated eigenvector solutions:

$$\mathbf{x}_2(t) = e^t[1, 0, 2, 0]^T$$

$$\mathbf{x}_3(t) = e^t[1, -2, 0, 2]^T$$



Example 2 (cont.)

To find 4th solution check B^2 :

$$B^2 = \begin{bmatrix} -8 & -8 & 4 & -4 \\ 0 & 0 & 0 & 0 \\ -16 & -16 & 8 & -8 \\ 8 & 8 & -4 & 4 \end{bmatrix}$$

$\Rightarrow RREF(B^2)$ has only one nonzero row $[1, 1, -1/2, 1/2]$.

Construct basis of $\text{null}(B^2)$ by setting
 $x_2, x_3 = 0, x_4 = 2 \rightarrow \mathbf{u}_1 = [-1, 0, 0, 2]^T$
 $x_2, x_4 = 0, x_3 = 2 \rightarrow \mathbf{u}_2 = [1, 0, 2, 0]^T$
 $x_3, x_4 = 0, x_2 = 1 \rightarrow \mathbf{u}_3 = [1, -1, 0, 0]^T$

Check which are *not* eigenvectors:

$$B\mathbf{u}_1 = -2\mathbf{v}_2, \quad B\mathbf{u}_2 = \mathbf{0}, \quad B\mathbf{u}_3 = \mathbf{v}_2$$

\Rightarrow Can choose $\mathbf{v}_4 = \mathbf{u}_1$ (simple).

Associated solution:

$$\begin{aligned} \mathbf{x}_4(t) &= e^t(I\mathbf{v}_4 + tB\mathbf{v}_4) = e^t(\mathbf{u}_1 - 2t\mathbf{v}_2) \\ &= e^t[-1 - 2t, 0, -4t, 2]^T \end{aligned}$$

