Math 3331 Differential Equations

9.8 Higher Order Linear Equations

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- Equivalent System
- Examples
- Homogeneous ODE
- Superposition Principle
- Homogeneous Eqns with Constant Coefficients
- 2nd Order Equations: Examples
- Higher Order Equations: Examples

Equivalent System

Form: $y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = F(t)$ (1)

Equivalent 1st order system:

$$x_1 = y,\ x_2 = y',\ \ldots,\ x_n = y^{(n-1)}$$

$$\Rightarrow \begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= -a_n(t)x_1 - \cdots - a_1(t)x_n \\ &\quad +F(t) \end{aligned}$$

or $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ (2)

where $A(t) =$

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \cdots & -a_1(t) \end{bmatrix}$$

$$\mathbf{f}(t) = [0, 0, \ldots, 0, F(t)]^T$$

$\mathbf{x}(t)$ is solution of (2) iff

$$\begin{aligned} \mathbf{x}(t) &= [x_1(t), x_2(t), \ldots, x_n(t)]^T \\ &= [y(t), y'(t), \ldots, y^{(n-1)}(t)]^T \end{aligned}$$

and $y(t)$ is solution of (1).

Initial Value Problem:

$\mathbf{x}(t_0) = \mathbf{x}_0 = [y_0, y_0', \ldots, y_0^{(n-1)}]^T \Leftrightarrow$

$y^{(j)}(t_0) = y_0^{(j)}$, $0 \le j \le n-1$ (3)

Thm: *(Existence/Uniqueness)*
Assume $a_1(t), \ldots, a_n(t), F(t)$ are continuous on an interval I and $t_0 \in I$. Then (1) with IC (3) has a unique solution on I for any values of $y_0, y_0', \ldots, y_0^{(n-1)}$.

Example 1

Ex.: $my'' + \mu y' + ky = F(t)$, or

$$y'' + (\mu/m)y' + (k/m)y = F(t)/m$$

Equivalent system: $x_1 = y$, $x_2 = y' \Rightarrow$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -\mu/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F(t)/m \end{bmatrix}$$

Example 2

Ex.: $y''' + a_1 y'' + a_2 y' + a_3 y = F(t) \Rightarrow$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \ \mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix}$$

Homogeneous ODE

Homogeneous ODE: $y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = 0$ (4)

Def.: (*Linear Dependence and Independence of Functions*)
n functions $y_1(t), \ldots, y_n(t)$ are linearly dependent on an interval I if there are constants $c_1, \ldots, c_n$, not all zero,

s.t. $c_1y_1(t) + \ldots + c_ny_n(t) = 0$

for all $t \in I$. They are linearly independent on I if they are not linearly dependent.

Criterion for L.I.: Let $W(t) =$

$$\det \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix}$$

be the *Wronskian* of $y_1(t), \ldots, y_n(t)$. If $W(t_0) \neq 0$ for some $t_0 \in I$, then $y_1(t), \ldots, y_n(t)$ are linearly independent on I.

Superposition Principle

Homogeneous ODE: $y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = 0$ (4)

Superposition Principle:

If $y_1(t), \ldots, y_k(t)$ are solutions of (4), then any linear combination

$$c_1y_1(t) + c_2y_2(t) + \cdots + c_ky_k(t)$$

is also a solution.

Ex.: $y_1(t) = \cos t,\ y_2(t) = \sin t \Rightarrow$

$$W(t) = \begin{vmatrix} \cos t(t) & \sin t \\ -\sin t(t) & \cos t \end{vmatrix} = 1$$

Main Thm.:

Let $y_1(t), \ldots, y_n(t)$ be n solutions of (4) on an interval I on which $a_1(t), \ldots, a_n(t)$ are continuous.

(a) If $W(t_0) = 0$ for some $t_0 \in I$, then $y_1(t), \ldots, y_n(t)$ are linearly dependent on I.

(b) If $W(t_0) \neq 0$ for some $t_0 \in I$, then $W(t) \neq 0$ for all $t \in I$, and

$$y(t) = c_1y_1(t) + \cdots + c_ny_n(t)$$

is a general solution of (4).

(c) If $y_p(t)$ is a particular solution of (1) and $y(t)$ is a general solution of (4), then $y(t) + y_p(t)$ is a general solution of (1).

Homogeneous Eqns with Constant Coefficients

Homogeneous ODE: $y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = 0$ (4)

Def.: A fundamental set of solutions (F.S.S.) for (4) is a set of linearly independent solutions $y_1(t), \ldots, y_n(t)$.

Constant Coefficients Case:
($a_j =$ const)

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \quad (5)$$

Try solution $y(t) = e^{\lambda t} \Rightarrow$

$$p(\lambda) \equiv \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0 \quad (6)$$

Def.: (6) is called the characteristic equation of (5).

Associated Linear System:

$$\mathbf{x}' = A\mathbf{x} \qquad (7)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

Thm.:
(a) The characteristic polynomial of A is $(-1)^n p(\lambda)$, i.e.: the eigenvalues of A are the roots of p.

Homogeneous Eqns with Constant Coefficients

Constant Coefficients Case: ($a_j =$ const)

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \quad (5)$$

Try solution $y(t) = e^{\lambda t} \Rightarrow$

$$p(\lambda) \equiv \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0 \quad (6)$$

Def.: (6) is called the characteristic equation of (5).

Thm.:

(b1) If λ is a real root of p of multiplicity m, then (5) has the linearly independent solutions

$$e^{\lambda t},\ te^{\lambda t},\ \ldots,\ t^{m-1}e^{\lambda t}$$

Thm.:

(b2) If $\lambda = \alpha + i\beta$ is a complex root of p of multiplicity m, then (5) has the linearly independent solutions

$$e^{\alpha t}\cos\beta t,\ te^{\alpha t}\cos\beta t,\ \ldots,\ t^{m-1}e^{\alpha t}\cos\beta t$$
$$e^{\alpha t}\sin\beta t,\ te^{\alpha t}\sin\beta t,\ \ldots,\ t^{m-1}e^{\alpha t}\sin\beta t$$

(c) The collection of functions obtained in (b) if λ runs through all roots, with only one root of a complex pair considered, is a F.S.S. for (5).

(d) If λ is any (real or complex) root of p, then $[1, \lambda, \ldots, \lambda^{(n-1)}]^T$ is a basis of null$(A-\lambda I)$, i.e.: the geometric multiplicity of any eigenvalue of A is 1.

2nd Order Equations: Example 1

Ex.: $y'' - 3y' + 2y = 0$
IC: $y(0) = 2,\ y'(0) = 1$

$$p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

roots: $\lambda_1 = 2 \rightarrow y_1(t) = e^{2t}$
$\lambda_2 = 1 \rightarrow y_2(t) = e^{t}$

General solution: $y(t) = c_1 e^{2t} + c_2 e^{t}$

Match c_1, c_2 to IC:

$$\begin{array}{rcccl} y(0) & = & c_1 + c_2 & = & 2 \\ y'(0) & = & 2c_1 + c_2 & = & 1 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\Rightarrow y(t) = -e^{2t} + 3e^{t}$$

2nd Order Equations: Example 2

Ex.: $y'' + 2y' + 2y = 0$

IC: $y(0) = 2,\ y'(0) = 3$

$$p(\lambda) = \lambda^2 + 2\lambda + 2 = (\lambda + 1)^2 + 1$$

roots: $\lambda = 1 + i,\ \overline{\lambda}$

$\Rightarrow y_1(t) = e^{-t} \cos t,\ y_2(t) = e^{-t} \sin t$

General solution:

$$y(t) = e^{-t}(c_1 \cos t + c_2 \sin t)$$

Match c_1, c_2 to IC:

$$\begin{array}{rcccl} y(0) & = & c_1 & = & 2 \\ y'(0) & = & -c_1 + c_2 & = & 3 \quad \Rightarrow \quad c_2 = 5 \end{array}$$

$$\Rightarrow y(t) = e^{-t}(2 \cos t + 5 \sin t)$$

2nd Order Equations: Example 3

Ex.: $y'' + 2y' + y = 0$
IC: $y(0) = 2,\ y'(0) = -1$

$$p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$$

double root $\lambda = -1 \Rightarrow$ F.S.S.: e^{-t}, te^{-t}

General solution: $y(t) = e^{-t}(c_1 + c_2 t)$

$\Rightarrow y_1(t) = e^{-t}\cos t,\ y_2(t) = e^{-t}\sin t$
General solution:

$$y(t) = e^{-t}(c_1 \cos t + c_2 \sin t)$$

Match c_1, c_2 to IC:

$$\begin{array}{rcccl} y(0) & = & c_1 & = & 2 \\ y'(0) & = & -c_1 + c_2 & = & 3 \quad \Rightarrow \quad c_2 = 5 \end{array}$$

$$\Rightarrow y(t) = e^{-t}(2\cos t + 5\sin t)$$

Higher Order Equations: Example 1

Ex.: $y''' + 6y'' + 11y' + 6y = 0$

$$p(\lambda) = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = (\lambda+1)(\lambda+2)(\lambda+3)$$

$\Rightarrow$ roots: $-1, -2, -3$

$\Rightarrow$ F.S.S.: e^{-t}, e^{-2t}, e^{-3t}

$\Rightarrow$ General solution:

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}$$

Higher Order Equations: Example 2

Ex.: $y''' - y' = 0$

$$\begin{aligned} p(\lambda) &= \lambda^3 - \lambda = \lambda(\lambda^2 - 1) \\ &= \lambda(\lambda - 1)(\lambda + 1) \end{aligned}$$

$\Rightarrow$ roots: $0,\ 1,\ -1$

$\Rightarrow$ F.S.S.: $1,\ e^t,\ e^{-t}$

Higher Order Equations: Example 3

Ex.: $y'''' - 2y''' + 2y' - y = 0$

$$\begin{aligned} p(\lambda) &= \lambda^4 - 2\lambda^3 + 2\lambda - 1 \\ &= (\lambda - 1)^3(\lambda + 1) \end{aligned}$$

$\Rightarrow$ roots: 1 $(m = 3)$, -1 $(m = 1)$

$\Rightarrow$ F.S.S.: e^t, te^t, t^2e^t, e^{-t}

$\Rightarrow$ General solution:

$$y(t) = e^{-t}(c_1 + c_2t + c_3t^2) + c_4e^{-t}$$

Higher Order Equations: Example 4

Ex.: $y'''' + 4y''' + 14y'' + 20y' + 25y = 0$

$$\begin{aligned} p(\lambda) &= \lambda^4 + 4\lambda^3 + 14\lambda^2 + 20\lambda + 25 \\ &= (\lambda^2 + 2\lambda + 5)^2 \end{aligned}$$

$\Rightarrow \lambda^2 + 2\lambda + 5 = (\lambda + 1)^2 + 4 = 0$

$\Rightarrow$ roots: $-1 \pm 2i$ $(m = 2)$

$\Rightarrow$ F.S.S.: $e^{-t}\cos 2t$, $te^{-t}\cos 2t$

$e^{-t}\sin 2t$, $te^{-t}\sin 2t$