1. Find the solution of the following initial-value problem

   1. $\frac{dy}{dt} = ty^2, \quad y(0) = 1.$
   2. $\frac{dy}{dt} = ry + a, \quad y(0) = y_0. \quad (r,\, a,\, y_0 \text{ parameters})$
   3. $\frac{dy}{dt} = \frac{y}{t}, \quad y(1) = -2.$
   4. $\frac{dy}{dt} = \frac{\sin t}{y}, \quad y(\frac{\pi}{2}) = 1.$
   5. $\frac{dy}{dt} = 1 + y^2, \quad y(0) = 1.$
   6. $\frac{dy}{dt} = \frac{t}{y}, \quad y(0) = 1.$
   7. $\frac{dy}{dt} + \cos ty = \frac{1}{2}\sin 2t, \quad y(0) = 1.$
   8. $\frac{dy}{dt} + 2ty = 2t^3, \quad y(0) = -1.$
   9. $\frac{dy}{dt} + \frac{y}{1+t} = 2, \quad y(0) = 1.$
   10. $\frac{dy}{dt} - \frac{n}{t}y = e^t t^n, \quad y(1) = 1.$

2. A tank initially contains 50 gal of sugar water having a concentration of 2 lb of sugar for each gal of water. At time zero, pure water begins pouring into the tank at a rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank so that the volume sugar-waater solution in the tank remains constant.

   (a) How much sugar is in tank after 10 minutes?

   (b) How long will it take the sugar content in the tank to dip below 20 lb?

   (c) What will be the eventual sugar content in the tank?

3. Suppose that y is a solution to the initial value problem

   $$y' = (y^2 - 1)e^{ty} \text{ and } y(1) = 0.$$

   Show that $-1 < y(t) < 1$ for all t for which y is defined.

4. Suppose a population is growing according to the logistic equation

   $$\frac{dP}{dt} = f(P) \quad \text{where } f(P) = r_0\left(1 - \frac{P}{K}\right)P$$

   with r_0 being the natural reproductive rate and K being the carrying capacity. Perform each of the following tasks without the aid of technology.

(i) Sketch a graph of $f(P)$

(ii) Use the graph of f to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

(iii) Sketch the equilibrium solutions in the t-P plans. These equilibrium solutions divide the t-P plane into regions. Sketch at least one solution trajectory in each of these regions.

5. Find the solution of the following initial-value problem

   a. $y'' - 3y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1.$

   b. $y'' + 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 3.$

   c. $y'' - 2y' + y = 0, \quad y(0) = 2, \quad y'(0) = -1.$

6. A 0.1 kg mass is attached to a spring having a spring constant 3.6 kg/s. The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present, find the amplitude, frequency, and phase of the resulting motion. Plot the solution.

7. Find the solution of the following initial-value problems

   a. $y'' + 3y' + 2y = 3e^{-4t}, \quad y(0) = 1, \quad y'(0) = 0.$

   b. $y'' + 2y' + 2y = 2\cos 2t, \quad y(0) = -2, \quad y'(0) = 0.$

   c. $y'' - 2y' + y = t^3, \quad y(0) = 1, \quad y'(0) = 0.$

   d. $y'' + 4y' + 4y = 2e^{-2t}, \quad y(0) = 0, \quad y'(0) = 1.$

8. Verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2 y'' + 3ty' - 3y = 0, \qquad \text{for } t > 0.$$

   Use the variation of parameters to find the general solution to

$$t^2 y'' + 3ty' - 3y = \frac{1}{t}, \qquad \text{for } t > 0.$$

9. An undamped spring-mass system with external driving force is modeled with

$$x'' + 25x = 4\cos 5t.$$

   The parameters of this equation are “tuned” so that the frequency of the driving force equals the natural frequency of the undriven system. Suppose that the mass is displaced one positive unit and released from rest.

   (a) Find the position of the mass as a function of time. What part of the solution guarantees that this solution resonates?

   (b) Sketch the solution found in part (a).