1. Find the Laplace Transform of the following function

$$f(x) = \begin{cases} \sin x, & 0 \le x < \pi/2 \\ 2\cos x + 1, & x \ge \pi/2 \end{cases}.$$

Solution. We will rewrite the function $f(x)$ using the Heaviside function. We have:

$$\begin{aligned} f(x) &= \sin(x)H_{0\frac{\pi}{2}} + (2\cos(x)+1)H_{\frac{\pi}{2}} \\ &= \sin(x)\left[H(x) - H(x-\frac{\pi}{2})\right] + (2\cos(x)+1)\,H\left(x-\frac{\pi}{2}\right) \\ &= \sin(x)H(x) + \Big(-\sin(x) + 2\cos(x) + 1\Big)H\left(x-\frac{\pi}{2}\right) \end{aligned}$$

Note that

$$\sin(x) = \sin(x - \frac{\pi}{2} + \frac{\pi}{2}) = \cos(x-\frac{\pi}{2}) \text{ and } \cos(x) = \cos(x - \frac{\pi}{2} + \frac{\pi}{2}) = -\sin(x-\frac{\pi}{2}).$$

$$\begin{aligned} f(x) &= \sin(x)H(x) + \Big(-\sin(x) + 2\cos(x) + 1\Big)H\left(x-\frac{\pi}{2}\right) \\ &= \sin(x)H(x) + \Big(-\cos(x-\frac{\pi}{2}) - 2\sin(x-\frac{\pi}{2}) + 1\Big)H\left(x-\frac{\pi}{2}\right) \end{aligned}$$

Then, the Laplace Transform for f is

$$\begin{aligned} F(s) &= \mathcal{L}\left\{\sin(x)H(x) + \Big(-\cos(x-\frac{\pi}{2}) - 2\sin(x-\frac{\pi}{2}) + 1\Big)H\left(x-\frac{\pi}{2}\right)\right\}(s) \\ &= \frac{1}{s^2+1} - \frac{s}{s^2+1}\cdot e^{-\pi s/2} - \frac{2}{s^2+1}\cdot e^{-\pi s/2} + \frac{1}{s}\cdot e^{-\pi s/2}. \end{aligned}$$

◀

2. Let $F(s) = \dfrac{2+se^{-3s}}{s^2+2s}$ be the Laplace transformation of a piecewise defined function $f(t)$. Find the piecewise defined function.

Solution. Rewrite $F(s)$ as a partial fractions for which we know their inverse Lapalce Transforms:

$$\begin{aligned} F(s) &= \frac{2+se^{-3s}}{s^2+2s}F(s) = \frac{2}{s(s+2)} + \frac{se^{-3s}}{s(s+2)} \\ &= \frac{A}{s} + \frac{B}{s+2} + \frac{1}{s+2}\cdot e^{-3s} \quad A = 1, B = -1 \\ &= \frac{1}{s} - \frac{1}{s+2} + \frac{1}{s+2}\cdot e^{-3s} \end{aligned}$$

Then, by applying the inverse Laplace transform operator, we get

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+2} + \frac{1}{s+2}\cdot e^{-3s}\right\}(t) \\ &= 1 - e^{-2t} + e^{-2(t-3)}H(t-3) \end{aligned}$$

The function f can be written as

$$f(t) = \begin{cases} 1 - e^{-2t} & 0 \le t \le 3 \\ 1 - e^{-2t} + e^{-2(t-3)} & t > 3 \end{cases}.$$

◀

3. Use the Laplace Transform to find the solution of the following initial-value problems

   a. $y'' + y = \cos 2t, \quad y(0) = 0, \quad y'(0) = 1.$

   Solution. Let $Y(s) = \mathcal{L}(y(t))$. Then

$$\begin{aligned} &\mathcal{L}(y'' + y) = \mathcal{L}\{\cos 2t\} \quad \Rightarrow \quad \mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}\{\cos 2t\} \\ &\quad \Rightarrow \quad \big(s^2Y - sy(0) - y'(0)\big) + Y = \frac{s}{s^2+4} \\ &\text{I.Cs.} \quad \Rightarrow \quad (s^2+1)Y - 1 = \frac{s}{s^2+4} \quad \Rightarrow \quad Y(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1} \end{aligned}$$

   Note that the partial fraction decomposition of $Y(s)$ is

$$\begin{aligned} Y(s) &= \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1} \\ &= \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} + \frac{1}{s^2+1} \qquad A = 1/3, C = -1/3, B = D = 0 \\ &= \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1} \end{aligned}$$

   Then

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(Y(s))(t) \\ &= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{3}\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) \\ &= \frac{1}{3}\cos t - \frac{1}{3}\cos 2t + \sin t \end{aligned}$$

◀

   b. $y'' - y = e^t, \quad y(0) = 0, \quad y'(0) = 0.$

Solution. Let $Y(s) = \mathcal{L}(y(t))$. Then

$$\begin{aligned}
&\mathcal{L}(y'' - y) = \mathcal{L}\{e^t\} \quad \Rightarrow \quad \mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}\{e^t\} \\
&\Rightarrow \quad \left(s^2Y - sy(0) - y'(0)\right) - Y = \frac{1}{s-1} \\
&\text{I.Cs.} \quad \Rightarrow \quad (s^2-1)Y = \frac{1}{s-1} \quad \Rightarrow \quad Y(s) = \frac{1}{(s+1)(s-1)^2}
\end{aligned}$$

Note that the partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{1}{(s+1)(s-1)^2} = \frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}$$

Then

$$\begin{aligned}
y(t) = \mathcal{L}^{-1}(Y(s))(t) &= \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) \\
&= \frac{1}{4}e^{-t} - \frac{1}{4}e^t + \frac{1}{2}te^t
\end{aligned}$$

◀

c. $y'' + y = g(t), \quad y(0) = 0, \quad y'(0) = 1, \quad \text{where } g(t) = \begin{cases} 2t, & \text{for } 0 \le t < 1 \\ 2, & \text{for } 1 \le t < \infty \end{cases}$.

Solution. This is example 5.15 of Sec. 5.5, pg. 221. We have done similar one in class. ◀

4. Section 5.5: 4, 14

4. Since $f(t) = \sin t$ has transform

$$F(s) = \frac{1}{s^2+1},$$

$g(t) = H(t - \pi/6)\sin(t - \pi/6)$ has transform

$$\begin{aligned} G(s) &= e^{-\pi(s/6)}F(s) \\ &= e^{-\pi s/6}\frac{1}{s^2+1} \\ &= \frac{e^{-\pi s/6}}{s^2+1}. \end{aligned}$$

14. The function

$$f(t) = \begin{cases} 3, & \text{if } 0 \le t < 1; \\ 2, & \text{if } 1 \le t < 2; \\ 1, & \text{if } 2 \le t < 3; \\ 0, & \text{otherwise} \end{cases}$$

can be written as

$$\begin{aligned} f(t) &= 3H_{01}(t) + 2H_{12}(t) + H_{23}(t) \\ &= 3H(t) - 3H(t-1) + 2H(t-1) \\ &\quad - 2H(t-2) + H(t-2) - H(t-3) \\ &= 3H(t) - H(t-1) - H(t-2) \\ &\quad - H(t-3) \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= 3\,\mathcal{L}\{H(t)\}(s) - \mathcal{L}\{H(t-1)\}(s) \\ &\quad - \mathcal{L}\{H(t-2)\}(s) \\ &\quad - \mathcal{L}\{H(t-3)\}(s) \\ &= \frac{3}{s} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}. \end{aligned}$$

Section 5.5: 22, 25

22. A partial fraction decomposition.

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$
$$1 = A(s+2) + Bs$$

Then,

$$s = -2 \Rightarrow B = -\frac{1}{2}$$
$$s = 0 \Rightarrow A = \frac{1}{2}$$

and

$$\frac{1-e^{-s}}{s(s+2)} = \frac{1}{2s} - \frac{1}{2(s+2)} - \frac{e^{-s}}{2s} + \frac{e^{-s}}{2(s+2)}.$$

Note the transform pairs.

$$f(t) = 1 \leftrightarrow F(s) = \frac{1}{s}$$
$$g(t) = e^{-2t} \leftrightarrow G(s) = \frac{1}{s+2}$$

Thus

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1-e^{-s}}{s(s+2)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{2s} - \frac{1}{2(s+2)} - e^{-s}\left[\frac{1}{2s} - \frac{1}{2(s+2)}\right]\right\}(t) \\
&= \frac{1}{2}\big[\mathcal{L}^{-1}\{F(s)\}(t) - \mathcal{L}^{-1}\{G(s)\}(t) - \mathcal{L}^{-1}\{e^{-s}F(s)\}(t) + \mathcal{L}^{-1}\{e^{-s}G(s)\}(t)\big] \\
&= \frac{1}{2}\big[1 - e^{-2t} - H(t-1)f(t-1) + H(t-1)g(t-1)\big] \\
&= \frac{1}{2}\big[1 - e^{-2t} - H(t-1) + H(t-1)e^{-2(t-1)}\big]
\end{aligned}$$

25. Complete the square.

$$F(s) = \frac{2-e^{-2s}}{s^2+2s+2} = \frac{2-e^{-2s}}{(s+1)^2+1}.$$

Note the transform pair:

$$\sin t \Longleftrightarrow \frac{1}{s^2+1}.$$

Thus, by Proposition 2.12, we have another transform pair:

$$e^{-t}\sin t \Longleftrightarrow \frac{1}{(s+1)^2+1}.$$

Thus,

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{2-e^{-2s}}{(s+1)^2+1}\right\}(t) \\
&= 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}(t) - \mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{(s+1)^2+1}\right\}(t) \\
&= 2e^{-t}\sin t - H(t-2)e^{-(t-2)}\sin(t-2),
\end{aligned}$$

or, equivalently,

$$\begin{cases} 2e^{-t}\sin t, & 0 \le t < 2, \\ 2e^{-t}\sin t - e^{-(t-2)}\sin(t-2), & 2 \le t < \infty. \end{cases}$$

5. Consider the initial value problem

$$x' = -x + t, \quad 0 \le t \le 1, \quad x(0) = 0.5. \tag{1}$$

Use the Euler, RK2 and RK4 methods to approximate the value of $x(1)$ for a step size $h = 0.5$ and compute the error of your numerical solution.

Solution. We have $t_0 = 0$, $x_0 = 0.5$, and $f(t, y) = t - x$. Thus, the first step of Euler's method is completed as follows

$$\begin{aligned} x_1 &= x_0 + hf(t_0, x_0) = 0.5 + 0.5(0 - 0.5) = 0.25 \\ t_1 &= t_0 + h = 0 + 0.5 = 0.5 \end{aligned}$$

The second step follows

$$\begin{aligned} x_2 &= x_1 + hf(t_1, x_1) = 0.25 + 0.5 * (0.5 - 0.25) = 0.375 \\ t_2 &= t_1 + h = 0.5 + 0.5 = 1 \end{aligned}$$

The first step of RK2 method follows. First we compute the slopes

$$\begin{aligned} s_1 &= f(t_0, x_0) = f(0, 0.5) = 0 - 0.5 = -0.5 \\ s_2 &= f(t_0 + h, x_0 + hs_1) = f(0.5, 0.25) = 0.5 - 0.25 = 0.25 \end{aligned}$$

Youn can now update x and t

$$\begin{aligned} x_1 &= x_0 + h\frac{1}{2}(s_1 + s_2) = 0.5 + 0.5\frac{1}{2}(-0.5 + 0.25) = 0.4375 \\ t_1 &= t_0 + h = 0 + 0.5 = 0.5 \end{aligned}$$

The second iteration begins with computing the slopes

$$\begin{aligned} s_1 &= f(t_1, x_1) = f(0.5, 0.4375) = 0.5 - 0.4375 = 0.0625 \\ s_2 &= f(t_1 + h, x_1 + hs_1) = f(1, 0.46875) = 1 - 0.46875 = 0.53125 \end{aligned}$$

You can now update x and t

$$\begin{aligned} x_2 &= x_1 + h\frac{1}{2}(s_1 + s_2) = 0.4375 + 0.5\frac{1}{2}(0.0625 + 0.53125) = 0.5859375 \\ t_2 &= t_1 + h = 0.5 + 0.5 = 1 \end{aligned}$$

The first step of RK4 method follows. First we compute the four slopes

$$\begin{aligned} s_1 &= f(t_0, x_0) = f(0, 0.5) = -0.5 + 0 = -0.5 \\ s_2 &= f(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}s_1) = f(0.25, 0.375) = 0.25 - 0.375 = -0.125 \\ s_3 &= f(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}s_2) = f(0.25, 0.46875) = 0.25 - 0.46875 = -0.21875 \\ s_4 &= f(t_0 + h, x_0 + hs_3) = f(0.5, 0.390625) = 0.5 - 0.390625 = 0.109375 \end{aligned}$$

Youn can now update x and t

$$\begin{aligned} x_1 &= x_0 + h\frac{1}{6}(s_1 + 2(s_2+s_3)+s_4) \\ &= 0.5 + 0.5\frac{1}{6}(-0.5 + 2(-0.125 - 0.21875) + 0.109375) = 0.41015625 \\ t_1 &= t_0 + h = 0 + 0.5 = 0.5 \end{aligned}$$

The second iteration begins with computing the slopes

$$\begin{aligned} s_1 &= f(t_1, x_1) = f(0.5, 0.41015625) = 0.5 - 0.41015625 = 0.08984375 \\ s_2 &= f(t_1 + \frac{h}{2}, x_1 + \frac{h}{2}s_1) = f(0.75, 0.432617188) = 0.75 - 0.432617188 = 0.317382812 \\ s_3 &= f(t_1 + \frac{h}{2}, x_1 + \frac{h}{2}s_2) = f(0.75, 0.489501953) = 0.75 - 0.489501953 = 0.260498047 \\ s_4 &= f(t_1 + h, x_1 + hs_3) = f(1, 0.540405274) = 1 - 0.540405274 = 0.459594726 \end{aligned}$$

You can now update x and t

$$\begin{aligned} x_2 &= x_1 + h\frac{1}{6}(s_1 + 2(s_2+s_3)+s_4) \\ &= 0.41015625 + 0.5\frac{1}{6}(0.08984375 + 2(0.317382812 + 0.260498047) + 0.459594726) \\ &= 0.552256266 \\ t_2 &= t_1 + h = 0.5 + 0.5 = 1 \end{aligned}$$

The equation is linear and we can find its solution $x(t) = \frac{3}{2}e^{-t} + t - 1$ (note that $x(t) = x_h(t) + x_p(t)$ with $x_h(t) = ce^{-t}$ and $x_p(t) = t - 1$). We can compute the true values: $x(0.5) = \frac{3}{2}e^{-0.5} + 0.5 - 1 = 0.40979599$ and $x(1) = \frac{3}{2}e^{-1} = 0.551819162$ We can complete the following table

	time	approx.	true value	error
Euler	0.5	0.25	0.40979599	0.15979599
RK2	0.5	0.4375	0.40979599	0.02770401
RK4	0.5	0.41015625	0.40979599	0.00036026
Euler	1.0	0.375	0.551819162	0.176819162
RK2	1.0	0.5859375	0.551819162	0.034118338
RK4	1.0	0.552256266	0.551819162	0.000437104

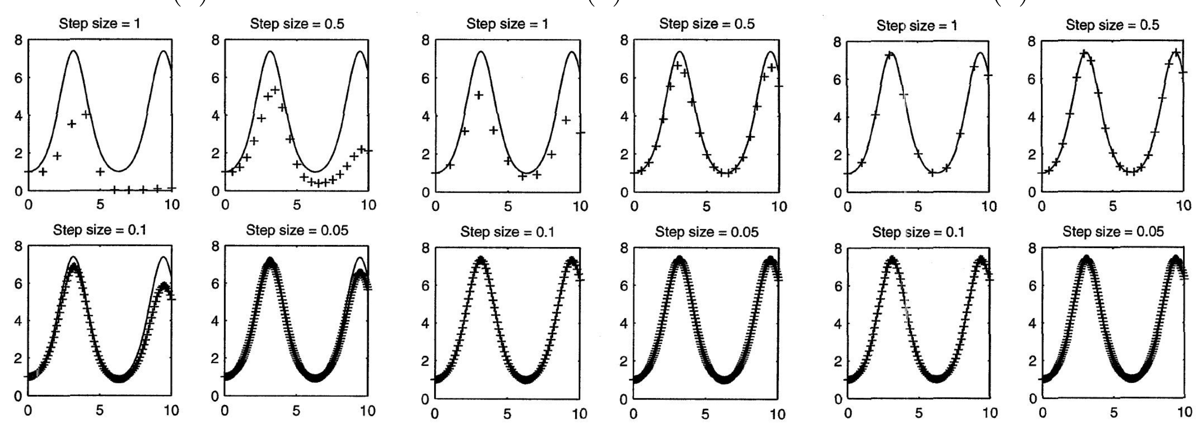
◀

6. Consider the initial value problem

$$x' = x \sin t, \quad t \geq 0, \quad x(0) = 1. \tag{2}$$

The equation is separable and the solution is $x(t) = e^{1-\cos t}$. The Euler method, RK2 and RK4 methods, with step sizes $h = 1$, 0.5, 0.1 and 0.05 produce the following results. Indicate each graph (1,2,3) by its corresponding numerical method and explain your answer.

(1) (2) (3)

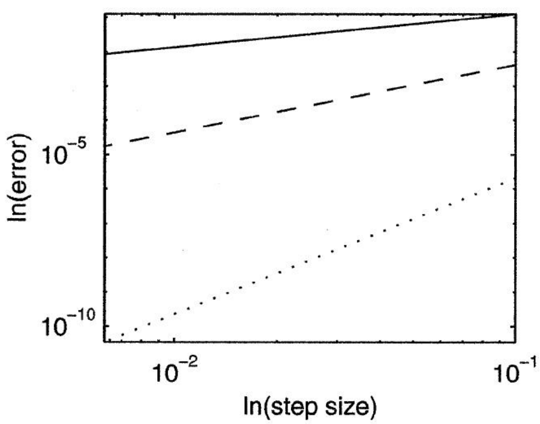


Solution. Euler's method , with step sizes $h = 1$, 0.5, 0.1 and 0.05, produces the results shown in Graph 1, RK2 method does a little better producing the results in Graph 2, RK4 method is the most accurate, producing the results in Graph 3. ◀

7. Consider the initial value problem

$$x' = x, \quad 0 \le t \le 1, \quad x(0) = 1. \tag{3}$$

The equation is separable and the solution is $x(t) = e^t$. We used the Euler method, RK2 and RK4 methods to compute the value of $x(1)$ and constructed a plot of the logarithm of the error versus the logarithm of the step size for each numerical method. The slope of the solid line is 0.9716, the slope of the dashed line is 1.9755, and the slope of the dotted line is 3.9730. Indicate each line by its corresponding numerical method and explain your answer.



Solution. Euler's method produces the results shown in the solid line, as the slope of the solid line is 0.9716, which is closed to 1, consistent with the fact that the Euler method is a first order algorithm. RK2 method produces the results shown in the dashed line as the slope of the dashed line is 1.9755, which is close to 2, consistent with the fact that RK2 is a second order method. Finally, RK4 method produces the results shown in the dotted line as the slope of the dotted line is 3.9730, which is close to 4, consistent with the fact that RK4 is a fourth order method. ◀

8. Write each initial value problems as a system of the first-order equations using vector notation.

a. $x'' + \delta x' - x + x^3 = \gamma \cos \omega t, \quad x(0) = x_0,\ x'(0) = v_0$

b. $x'' + \mu(x^2 - 1)x' + x = 0, \quad x(0) = x_0,\ x'(0) = v_0$

Solution. (a) With

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

we have

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -\delta u_2 + u_1 - u_1^3 + \gamma \cos \omega t, \end{aligned}$$

with initial conditions

$$u(0) = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

(b) With

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

we have

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= -\mu(u_1^2 - 1)u_2 - u_1, \end{aligned}$$

with initial conditions

$$u(0) = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

◀

9. Consider the predator-prey system

$$\begin{aligned} F' &= +0.2F - 0.1FS \\ S' &= -0.3S + 0.1FS \end{aligned}$$

Describe the behaviour of the system.
What happens to the solution that starts with $F(0) = 3$ and $S(0) = 2$?

Solution. If you plot a vector field of the system, you will notice that solutions appear to be closed, indicating a periodic oscillation of the two species, which eventually return to the starting levels.
A quick observation shows that for $(3, 2)$ we have $F' = 0, S' = 0$. Therefore, the solution starting at $(3, 2)$ remains at $(3, 2)$ for all time. This solution is called an equilibrium solution. ◀

10. Consider the system

$$\begin{aligned} x' &= 4x - 4x^2 - xy \\ y' &= 4y - xy - 2y^2 \end{aligned} .$$

Plot the nullclines for each equation of the given system. Calculate the equilibrium points and plot them in your sketch.

Solution. We did a similar one in class. Just coefficients are different. ◀

11. Consider the system

$$\begin{aligned} x' &= 1 - (y - \sin(x))\cos(x) \\ y' &= \cos(x) - y + \sin(x) \end{aligned} .$$

(i) Show that $x(t) = t, \quad y(t) = \sin(t)$ is a solution.

Solution. If $x(t) = t$ and $y(t) = \sin(t)$, then

$$x' = t' = 1$$

and

$$1 - (y - \sin(x))\cos(x) = 1 - (\sin(t) - \sin(t))\cos(t) = 1,$$

so the first equation is satisfied. Further,

$$y' = (\sin(t)' = \cos(t),$$

and

$$\cos(x) - y + \sin(x) = \cos(t) - \sin(t) + \cos(t) = \cos(t),$$

so the second equation is satisfied. ◀

(ii) Plot the solution in a phase plane.

Solution. You have to graph the function $y = \sin(x)$ in the phase plane. ◀

(iii) Consider the solution to the system with initial conditions $x(0) = \pi/2$ and $y(0) = 0$. shaow that $y(t) < \sin(x(t))$ for all t.

Solution. Because of the uniqueness of the solutions, the solution curve with initial conditions $x(0) = \pi/2$ and $y(0) = 0$ cannot cross the solution curve $x(t) = t, \quad y(t) = \sin(t)$ found in part(a). Thus, this new solution curve must remain below the solution in part(a) for all time. Therefore, if $(x(t), y(t))$ denotes the second solution, we must have $y(t) < \sin(x(t))$ for all time, as we showed in class. ◀

12. Section 8.3: 2, 5, 7, 10

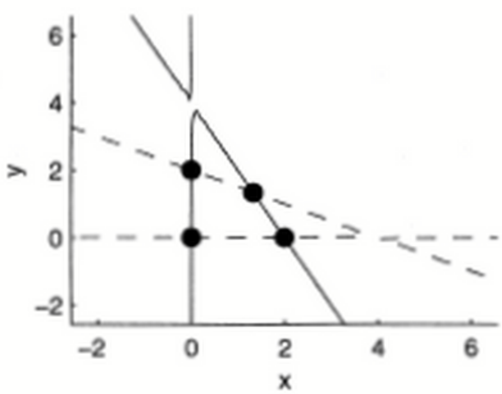
2. Set the right-hand side of $x' = 4x - 2x^2 - xy$ equal to zero.

$$4x - 2x^2 - xy = 0$$
$$x(4 - 2x - y) = 0$$

Thus, $x = 0$ and $4 - 2x - y = 0$ are the x-nullclines. They appear in a solid line style in the figure. Set the right-hand side of $y' = 4y - xy - 2y^2$ equal to zero.

$$4y - xy - 2y^2 = 0$$
$$y(4 - x - 2y) = 0$$

Thus, $y = 0$ and $4 - x - 2y = 0$ are the y-nullclines. They appear in a dashed line style in the figure.



The equilibrium points appear where the x-nullclines intersect the y-nullclines. These are $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(4/3, 4/3)$.

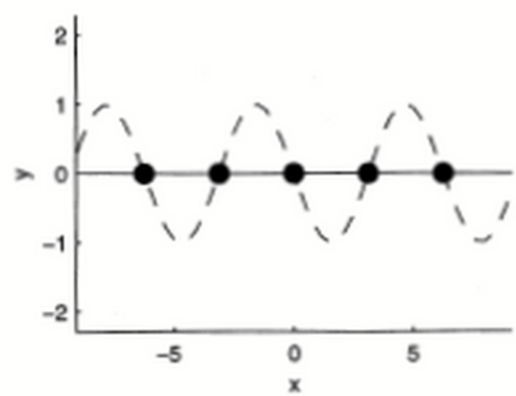
5. Set the right hand side of $x' = y$ equal to zero.

$$y = 0$$

Thus, $y = 0$ is the x-nullcline. It appears in a solid line style in the figure. Set the right hand side of $y' = -\sin x - y$ equal to zero.

$$-\sin x - y = 0$$
$$y = -\sin x$$

Thus, $y = -\sin x$ is the y-nullcline. It appears in a dashed line style in the figure.



The equilibrium points appear where the x-nullcline intersects the y-nullcline. These occur at the points $(k\pi, 0)$, where k is an integer. A few are shown in the figure.

12. Section 8.3: 2, 5, 7, 10

7. (a) If $x(t) = t$ and $y(t) = \sin t$, then

$$x' = (t)' = 1,$$

and

$$1-(y-\sin x)\cos x = 1-(\sin t-\sin t)\cos t = 1,$$

so the first equation is satisfied. Further,

$$y' = (\sin t)' = \cos t,$$

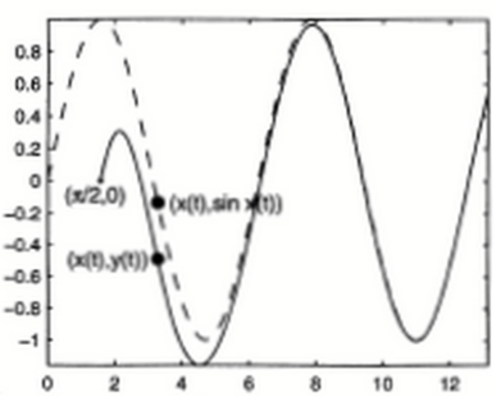
and

$$\cos x - y + \sin x = \cos t - \sin t + \sin t = \cos t,$$

so the second equation is satisfied.

(b) See the figure in part (c).

(c) Because of uniqueness, the solution with initial condition $x(0) = \pi/2$, $y(0) = 0$, cannot cross the solution $x = t$, $y = \sin t$ found in part (a). Thus, it must remain below the solution in part (a) for all time. Therefore, if $(x(t), y(t))$ denotes the second solution, we must have $y(t) < \sin x(t)$ for all time, as shown in the figure.



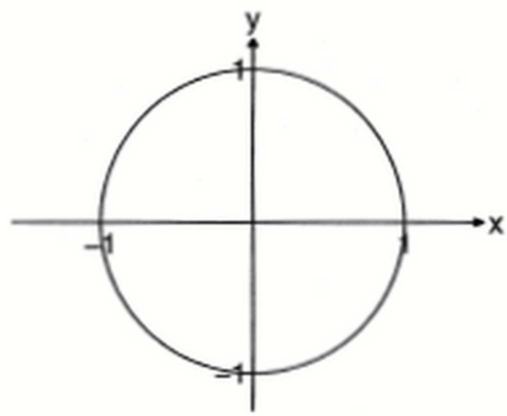
10. (a) Notice that if $x(t) = \sin t$ and $y(t) = \cos t$, then $x^2 + y^2 = 1$. Therefore we have

$$x' = \cos t \quad \text{and}$$
$$y - x(x^2 + y^2 - 1) = y = \cos t.$$

Next

$$y' = -\sin t \quad \text{and}$$
$$-x - y(x^2 + y^2 - 1) = -x = -\sin t.$$

(b) The solution curve in part (a) is the unit circle and is plotted below.



(c) The point $(x(0), y(0)) = (0.5, 0)$ is inside the unit circle, which is the solution curve from part (a). By the uniqueness theorem the solution curve starting at $(0.5, 0)$ cannot cross the unit circle. It must therefore stay inside the unit circle for all time. Hence $x^2(t) + y^2(t) < 1$ for all t.