Name and ID: ____________________

1. Find the solution of the initial-value problem

   (a)

$$\begin{aligned} x' &= 2x + 4y + 4z \\ y' &= x + 2y + 3z \\ z' &= -3x - 4y - 5z \end{aligned}$$

with $x(0) = 1$, $y(0) = -1$ and $z(0) = 0$.

Solution. In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 2 & 3 \\ -3 & -4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigen-pairs of $A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 2 & 3 \\ -3 & -4 & -5 \end{pmatrix}$ are

$$\begin{aligned} &\lambda_1 = -1, \quad \lambda_2 = 2i, \quad \lambda_3 = -2i \\ &v_1 = (0, -1, 1)^T, \quad v_2 = (-2, -1 - i, 2)^T, \quad v_3 = (-2, -1 + i, 2)^T. \end{aligned}$$

Using Euler's formula

$$\begin{aligned} w(t) &= e^{2it} \begin{pmatrix} -2 \\ -1 - i \\ 2 \end{pmatrix} = (\cos 2t + i \sin 2t) \left(\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right) \\ &= \left(\cos 2t \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right) + i \left(\cos 2t \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \right) \end{aligned}$$

The real and imaginary parts of w are solutions and we can write the general solution

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2\cos 2t \\ -\cos 2t + \sin 2t \\ 2\cos 2t \end{pmatrix} + c_3 \begin{pmatrix} -2\sin 2t \\ -\cos 2t - \sin 2t \\ 2\sin 2t \end{pmatrix}$$

If $x(0) = 1$, $y(0) = -1$ and $z(0) = 0$, then

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

We find that $c_1 = 1$, $c_2 = -1/2$, and $c_3 = 1/2$. Hence the solution is

$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} &= e^{-t}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -2\cos 2t \\ -\cos 2t + \sin 2t \\ 2\cos 2t \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2\sin 2t \\ -\cos 2t - \sin 2t \\ 2\sin 2t \end{pmatrix} \\ &= \begin{pmatrix} \cos 2t - \sin 2t \\ -e^{-t} - \sin 2t \\ e^{-t} - \cos 2t + \sin 2t \end{pmatrix}\end{aligned}$$

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(b)

$$\begin{aligned} x' &= 6x - 4z \\ y' &= 8x - 2y \\ z' &= 8x - 2z \end{aligned}$$

with $x(0) = -2$, $y(0) = -1$ and $z(0) = 0$.

Solution. In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 6 & 0 & -4 \\ 8 & -2 & 0 \\ 8 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigen-pairs of $A = \begin{pmatrix} 6 & 0 & -4 \\ 8 & -2 & 0 \\ 8 & 0 & -2 \end{pmatrix}$ are

$$\begin{aligned} &\lambda_1 = -2, \quad \lambda_2 = 2 + 4i, \quad \lambda_3 = 2 - 4i \\ &v_1 = (0, 1, 0)^T, \quad v_2 = (1 + i, 2, 2)^T, \quad v_3 = (1 - i, 2, 2)^T. \end{aligned}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{-2t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} \cos 4t - \sin 4t \\ 2\cos 4t \\ 2\cos 4t \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} \cos 4t + \sin 4t \\ 2\sin 4t \\ 2\sin 4t \end{pmatrix}$$

If $x(0) = -2$, $y(0) = -1$ and $z(0) = 0$, then

$$\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We find that $c_1 = -1$, $c_2 = 0$, and $c_3 = -2$. Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = -e^{-2t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 2e^{2t}\begin{pmatrix} \cos 4t + \sin 4t \\ 2\sin 4t \\ 2\sin 4t \end{pmatrix} = \begin{pmatrix} -2e^{2t}(\cos 4t + \sin 4t) \\ -e^{-2t} - 4e^{2t}\sin 4t \\ -4e^{2t}\sin 4t \end{pmatrix}$$

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(c)

$$\begin{aligned} x' &= -4x + 8y + 8z \\ y' &= -4x + 4y + 2z \\ z' &= 2z \end{aligned}$$

with $x(0) = 1$, $y(0) = 0$ and $z(0) = 0$.

Solution. In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -4 & 8 & 8 \\ -4 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigen-pairs of $A = \begin{pmatrix} -4 & 8 & 8 \\ -4 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ are

$$\begin{aligned} &\lambda_1 = 2, \quad \lambda_2 = 4i, \quad \lambda_3 = -4i \\ &v_1 = (0, -1, 1)^T, \quad v_2 = (1 - i, 1, 0)^T, \quad v_3 = (1 + i, 1, 0)^T. \end{aligned}$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \cos 4t + \sin 4t \\ \cos 4t \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -\cos 4t + \sin 4t \\ \sin 4t \\ 0 \end{pmatrix}$$

If $x(0) = 1$, $y(0) = 0$ and $z(0) = 0$, then

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

We find that $c_1 = 0$, $c_2 = 0$, and $c_3 = -1$. Hence the solution is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos 4t - \sin 4t \\ -\sin 4t \\ 0 \end{pmatrix}$$

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2. Find the general solution of the system

$$\begin{aligned} x' &= 6x - 5y + 10z \\ y' &= -x + 2y - 2z \\ z' &= -x + y - z \end{aligned}$$

Solution. The eigen-pairs of $A = \begin{pmatrix} 6 & -5 & 10 \\ -1 & 2 & -2 \\ -1 & 1 & -1 \end{pmatrix}$ are

$$\lambda_1 = 5, \quad \lambda_2 = \lambda_3 = 1 \qquad v_1 = (-5, 1, 1)^T, \quad v_2 = (1, 1, 0)^T, \quad v_3 = (-2, 0, 1)^T.$$

Proceed to the solution on your own! ◀

3. Find the general solution of the system

$$\begin{aligned} x' &= -2x + y - z \\ y' &= x - 3y \\ z' &= 3x - 5 \end{aligned}$$

Solution. Note that this linear system is **not** homogeneous, therefore we cannot solve it yet! You can skip it. Or if you like the challenge, substitute the last equation with $z' = 3x$ or $z' = 3x - 5y$ or $z' = 3x - 5z$. ◀

4. Classify the equilibrium point of the system $y' = Ay$. Sketch the phase portrait by hand.

(a) $A = \begin{pmatrix} -16 & 9 \\ -18 & 11 \end{pmatrix}$ (b) $A = \begin{pmatrix} 8 & 3 \\ -6 & -1 \end{pmatrix}$ (c) $A = \begin{pmatrix} -11 & -5 \\ 10 & 4 \end{pmatrix}$

(d) $A = \begin{pmatrix} 2 & -4 \\ 8 & 6 \end{pmatrix}$ (e) $A = \begin{pmatrix} 6 & -5 \\ 10 & -4 \end{pmatrix}$ (f) $A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$

Solution. Please practice how to solve 2×2 linear systems and how to classify them. In the exam you will be asked to solve one such linear system.
Also practice how to graph:
1. Nodal Source(both eigenvalues are positive and distinct),
2. Nodal Sink (both eigenvalues are negative and distinct),
3. Saddle Point (one eigenvalue is positive, the other negative),
4. Center (eigenvalues are complex of form $z = \pm bi$),
5. Spiral Source (eigenvalues are complex of form $z = a \pm bi$ with $a > 0$),
6. Spiral Sink (eigenvalues are complex of form $z = a \pm bi$ with $a < 0$).
You have to know how to explain!

(a) is SADDLE: To show why, you can either find the eigenvalues, which will be one positive, the other negative, hence the classification is done, or you can show as follows: If

$$A = \begin{pmatrix} -16 & 9 \\ -18 & 11 \end{pmatrix}$$

then the trace is $T = -5$ and the determinant is $D = -14 < 0$. Hence, the equilibrium point at the origin is a saddle. Hence the classification is done!

In order to sketch the SADDLE portrait, you solve it:

The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 + 5\lambda - 14$$

which produces eigenvalues $\lambda_1 = -7$ and $\lambda_2 = 2$. Because

$$A - \lambda_1 I = A + 7I = \begin{pmatrix} -9 & 9 \\ -18 & 18 \end{pmatrix} \to v_1 = (1, 1)^T$$

leading to the exponential solution

$$y_1(t) = e^{\lambda_1 t} v_1 = e^{-7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Because

$$A - \lambda_2 I = A - 2I = \begin{pmatrix} -18 & 9 \\ -18 & 9 \end{pmatrix} \to v_2 = (1, 2)^T$$

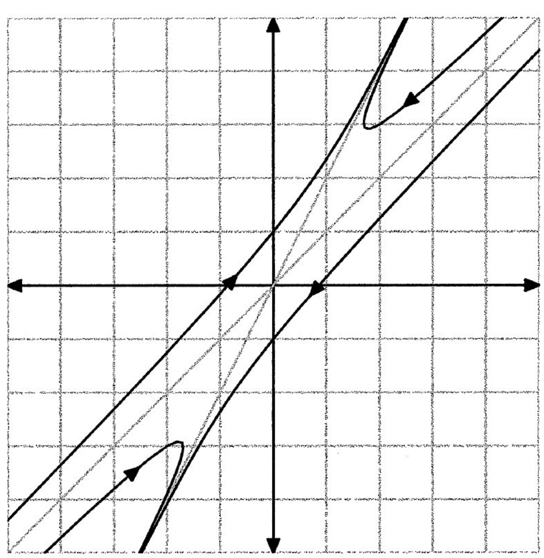
leading to the exponential solution

$$y_2(t) = e^{\lambda_2 t} v_2 = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solutions approach the halfline generated by $c_2(1, 2)^T$ as they move forward in time, but they approach the halfline generated by $c_1(1, 1)^T$ as they move backward in time. A hand sketch follows



The answers for the rest are:

(b)NODAL SOURCE

(c)NODAL SINK

(d)SPIRAL SINK, the motion is counterclockwise

(e)SPIRAL SOURCE, the motion is counterclockwise

(f)CENTER, the motion is clockwise.

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