Math 3331 Differential Equations

1.1 Differential Equation Models

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Welcome

- Welcome to Math 3331 – Ordinary Differential Equations
- Course Info, Syllabus, Lecture Notes, and Homework are posted online:

  `math.uh.edu/~blerina/teaching.html`

Click to Math. 3331 to go to our class' website.
Everything will be posted there accordingly.

What to Expect After the Course

You are expected to have three set of abilities and knowledge:

1. Pure Math: be able to solve equations and study the properties of solutions.
2. Applied Math: Know how to model a real world problem by ODEs
3. Computer: able to use MATLAB to study ODE numerically and visually.

1.1 Differential Equation Models

- Basic Idea of Using Differential Equations
- Motivating Examples
  - Newton's Second Law of mechanics
    - Newton's Universal Law of Gravitation
    - Newton's Model for the Motion of a Ball
    - Newton's Model of Planetary Motion
  - Population Growth Models
    - Exponential
    - Logistic
- Worked out Examples from Exercises:
  - 1.1, 1.2, 1.3, 1.4

The use of differential equations makes available to us the full power of CALCULUS.

CALCULUS becomes complete with the study of diff. equations

Basic Idea of Using Differential Equations

Recall: The derivative of a function f : $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ rate of change

Two Ways of Computing the Rate of Change

- In mathematics, the rate at which a quantity changes is the (1) derivative of that quantity, e.g.

$$\lim_{h \to 0} \frac{y(t+h) - y(t)}{h} = \frac{d\,y}{d\,t}, \quad \frac{d\,v}{d\,t} \quad \text{etc.}$$

- The second way of computing the rate of change comes from (2) the application, and is different from one application to another.

When these two ways of computing the rate of change are equated, we get a differential equation.

Newton's Second Law of mechanics

Newton's Second Law of mechanics

The force acting on a mass is equal to the rate of change of momentum with respect to time.

Newton's Contribution : Differentiation and Integration are inverse processes.

$$\frac{d\,mv}{d\,t} = F.$$

Recall from physics : Momentum of an object $I = m \cdot v$

m is the mass , v is the velocity.

Newton says :

$$F = \frac{dI}{dt} = \frac{d(\overset{\text{constant}}{m}v)}{dt} = m \cdot \underbrace{\frac{dv}{dt}}_{\text{acceleration}} = m \cdot a$$

$$\Rightarrow \boxed{F = m \cdot a}$$

Newton's Universal Law of Gravitation

Newton's Universal Law of Gravitation

Any body with mass M attracts any other body with mass m directly toward the mass M, with a magnitude proportional to the product of the two masses and inversely proportional to the square of the distance r separating them.

$$F = \frac{G\,M\,m}{r^2}, \quad G \text{ a universal constant.}$$

The force these two objects attract each other is proportional to product of masses and inverse proportional to square of distance between them.

$$F = \frac{G \cdot Mm}{r^2}.$$

Newton's Model for the Motion of a Ball

Model for the Motion of a Ball Near the Surface of the Earth

Let x be the distance the ball is above the earth. Then

acceleration

$$\frac{d^2 x}{d\,t^2} = -g, \quad \text{gravity constant}$$

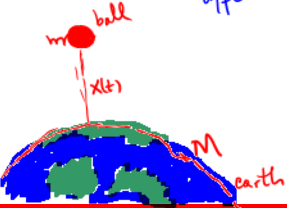
where the constant g is the earth's gravitational acceleration

Because of Newton's Law of Gravitation, we are able to model the motion of a ball affected by the gravity of earth:

By Newton: $F = \dfrac{G \cdot M \cdot m}{(r+x)^2}$

- r radius of earth, x distance (very very small)

M mass of earth



i.e $\quad F \approx \dfrac{G \cdot M}{r^2} \cdot m \quad$ (force with which the earth attracts the ball)

gravity constant $= g$

$$g \approx 32\, ft/s^2 = 9.8\, m/s^2$$

Hence, the ball is going up against gravity:

$$F = -\, m \cdot \underset{\substack{\uparrow \\ \text{acceleration}}}{a} = -\, m \cdot \frac{d^2x}{dt^2} = m \cdot g \Rightarrow \boxed{\frac{d^2x}{dt^2} = -g}$$

This shows that acceleration of free fall motion is <u>always</u> the gravity constant.

Newton's Model of Planetary Motion

Newton's Model of Planetary Motion

Let $\mathbf{x}(t)$ be the vector that gives the location of a planet relative to the sun. Then

$$m\frac{d^2\mathbf{x}}{d\,t^2} = -\frac{G\,M\,m}{|\mathbf{x}|^2}\,\frac{\mathbf{x}}{|\mathbf{x}|}.$$

Refer to book for more details.

Population Growth Models (I): Exponential

A Simple Model

The rate of change is proportional to the total population.

math. biology

$$\frac{dP}{dt} = r\,P, \qquad r \text{ the reproductive rate constant.}$$

Show that the exponential function is a solution:

$\frac{dP}{dt} = rP \implies$ $$P(t) = P_0\, e^{rt}.$$ solution for initial value of population.

Recall from Calculus: How to solve $\frac{dP}{dt} = rP$ given $P(0) = P_0$.

$\boxed{\frac{dP}{dt} = r \cdot P} \implies$ it can be rewritten as

$$\frac{dP}{P} = r \cdot dt$$ (because of definitions learned in Calculus I)

Integrate both sides: $\int \frac{dP}{P} = \int r\,dt \Rightarrow \ln P = r \cdot t + C$

$$\Rightarrow P(t) = e^{rt+c} = \underbrace{e^{c}}_{\text{constant}} \cdot e^{rt} = C \cdot e^{rt}$$

For $t=0$, $P(0) = C \cdot e^{r \cdot 0} = C = P_0 \Rightarrow \boxed{P(t) = P_0 \cdot e^{rt}}$

In Calculus, these types of equations were called "separable".

Population Growth Models (II): Logistic

A Better Model

A better model for the reproductive rate is $r(1 - P/K)$. Then

$$\frac{dP}{dt} = r\left(1 - P/K\right)P.$$

This ODE for $P(t)$, called the logistic equation, is much harder to solve, but it does a creditable job of predicting how single populations grow in isolated circumstances.

Depending on what you are working on, there is always place for better results.
"Logistic equation" is a better approach to study single populations in different situations!

Basic Ideas for Exercises

Basic Idea

The phase "y is proportional to x" implies that y is related to x via the equation

$$y = kx, \qquad \text{i.e.} \quad \frac{y}{x} = k \text{ is fixed}$$

where k is a constant.

In a similar manner,

- "y is proportional to the square of x" implies $y = kx^2$,
- "y is proportional to the product of x and z" implies $y = kxz$,
- "y is inversely proportional to the cube of x" implies $y = k/x^3$. i.e. $y \cdot x^3 = k$ is fixed!

In all exercises, use these ideas to model each application with a differential equation. All rates are assumed to be with respect to time.

Exercise 1.1

Example (Exercise 1.1)

The rate of growth of bacteria in a petri dish is proportional to the number of bacteria in the dish.

$$\frac{dy}{dt} = y'(t) = k \cdot y(t).$$

Exercise 1.2

Example (Exercise 1.2)

The rate of growth of a population of eld mice is inversely proportional to the square root of the population.

$$y'(t) = k \,/\, \sqrt{y(t)}.$$

Exercise 1.3

Example (Exercise 1.3)

A certain area can sustain a maximum population of 100 ferrets. The rate of growth of a population of ferrets in this area is proportional to the product of the population and the difference between the actual population and the maximum sustainable population.

$$y'(t) = k\, y(t)\,(100 - y(t)).$$

Exercise 1.4

Example (Exercise 1.4)

The rate of decay of a given radioactive substance is proportional to the amount of substance remaining.

$$y'(t) = -k\, y(t).$$