Math 3331 Differential Equations

2.1 Differential Equations and Solutions

Blerina Xhabli

Department of Mathematics, University of Houston

blerina@math.uh.edu
math.uh.edu/∼blerina/teaching.html

2.1 ODE and Solutions

- Definition of First Order ODE
  - Normal Form of ODE
- Solutions of ODE
  - General Solution and Solution Curves
  - Particular Solution
- Initial Value Problem
  - Solution of IVP
  - Interval of Existence
- Geometric Interpretation of ODE
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  - Geometric interpretation of Solutions
  - Numerical Solution of IVP
- Worked out Examples from Exercises:
  - 2.4, 2.13, 2.19

Recall from Calculus

- diff. eqn: $\frac{dP}{dt} = rP$, $P(0) = P_0$
- solution: $P(t) = P_0 \cdot e^{rt}$
- how does it work?

From calculus:

$y = e^x \Rightarrow y' = e^x = y$

$\Rightarrow y = e^{rx} \Rightarrow y' = r \cdot e^{rx} = ry$

we used this knowledge

$$P' = rP$$

$\Rightarrow$ rewrite as $\frac{P'}{P} = r$

$\Rightarrow$ integrate both sides $\int \frac{dP}{P} = \int r\,dt$

$\Rightarrow$ Solve $\Rightarrow$ $\ln P = rt + c \Rightarrow P(t) = e^{rt+c} = \underbrace{e^c}_{\text{constant}} \cdot e^{rt} = C e^{rt}$

$\Rightarrow$ use initial condition: $P(0) = C \cdot e^{r \cdot 0} = P_0 \Rightarrow C = P_0$

$\Rightarrow$ hence, $P(t) = P_0 \cdot e^{rt}$. Done.

Formal Definition of ODE

Definition of ODE

ODE is an equation involving an unknown function y of a single variable t together with one or more of its derivatives y', y'', etc.

First Order ODE: General (Implicit) Form

First order ODEs often arise naturally in the form

ex. $y' = ty \Rightarrow y' - ty = 0$ i.e $\phi(t, y, y') = y' - ty$

$$\phi(t, y, y') = 0,$$

Example

$$t + 4\, y\, y' = 0. \leftarrow \text{general}$$

This form is too general to deal with, and we will find it necessary to solve equation for y' to place it into "normal form"

$$y' = -\frac{t}{4y} \leftarrow \text{we prefer this}$$

ex . .

ODE

$$\begin{cases} y' = y + t^2 & \longleftarrow 1^{st} \text{ order} \\ y \cdot y'' + t^2 \cdot y = \cos t & \longleftarrow 2^{nd} \text{ order} \\ y' = \cos(ty) & \longleftarrow 1^{st} \text{ order} \\ y'' = y^2 & \longleftarrow 2^{nd} \text{ order} \end{cases}$$

Not an ODE

$$\frac{d^2 w}{dx^2} = r \cdot \frac{d^2 w}{dy^2} \quad , \quad \text{2 variables involved}$$

Normal Form of ODE

Normal Form

A first-order ODE of the form

$$y' = f(t, y)$$

is said to be in normal form.

leave "y'" in one side and the rest in the other side.

Examples

$$\left.\begin{aligned} y' &= y - t \\ y' &= -2\,t\,y \\ y' &= y^2 \\ y' &= \cos(t\,y) \end{aligned}\right\} \text{normal forms}$$

Example

Example

Place the first order ODE

$$y'^3 + y^2 = 1$$

into normal form.

Solution:

$$(y')^3 = 1 - y^2$$

$$\boxed{y' = \sqrt[3]{1-y^2}\,.}$$

Example

Example

Place the first order ODE

$$e^{y'} + y\,y' = 0$$

into normal form.

It can't be put in normal form.

Solutions of ODE

Solutions of ODE

A solution of the first-order ODE

$$y' = f(t, y)$$

is a differentiable function $y(t)$ such that

$$y'(t) = f(t, y(t))$$

for all t in the interval where $y(t)$ is defined.

Check Solutions: Example

Example

Show that $y(t) = t + 1 + Ce^t$ is a solution of

substitute y in the equation

$$y' = y - t.$$

Solution: To show it is true, we get the solution and substitute in eqn:

$$(t+1+C\cdot e^t)' = (\cancel{t}+1+C\cdot e^t) - \cancel{t}$$

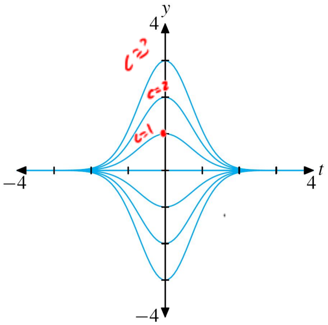
$$1+0+C\cdot e^t = 1+C\cdot e^t$$

✓

Yes, we're done.

General Solution and Solution Curves

To be continued



Example

Show that $y(t) = Ce^{-t^2}$ is a solution of

$$y' = -2\,t\,y$$

$-2tCe^{-t^2}$ $-2t\cdot C\cdot e^{-t^2}$

General Solution

The solution formula $\boxed{y(t) = Ce^{-t^2},}$ which depends on the arbitrary constant C, describes a family of solutions and is called a general solution.

Solution Curves

The graphs of these solutions, drawn in the figure, are called solution curves.

Particular Solution

Example

1. Show that $y(t) = 1/(C-t)$ is a general solution of
$$y' = y^2$$
2. Find a particular solution satisfying $y(0) = 1$.

$y(0) = \frac{1}{C-0} = 1$

$C = 1$

Given the value of the solution at a point, we can determine the unique particular solution.

Solve: $\frac{dy}{dt} = y^2 \iff \int \frac{1}{y^2}\, dy = \int dt \Rightarrow -\frac{1}{y} = t + C$

$\Rightarrow y(t) = \frac{-1}{t+C} = \boxed{\frac{1}{C-t}}$

Initial Value Problem

Initial Value Problem

A first-order ODE together with an initial condition,

$$y' = f(t, y), \quad y(t_0) = y_0$$

is called an initial value problem.

Solution of IVP

A solution of the IVP is a differentiable function $y(t)$ such that

1. $y'(t) = f(t, y(t))$ for all t in an interval containing t_0 where $y(t)$ is defined, and
2. $y(t_0) = y_0$

Example

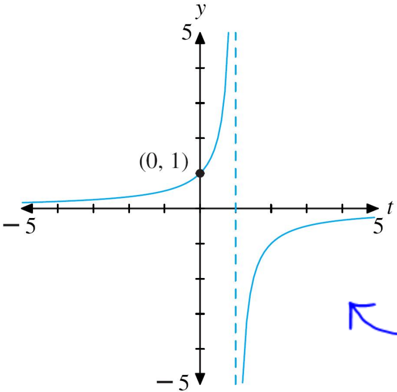
The function $y(t) = 1/(1-t)$ is the solution of the IVP

$$y' = y^2, \quad \text{with } y(0) = 1.$$

Interval of Existence

Interval of Existence

The interval of existence of a solution to an IVP is defined to be the largest interval over which the solution can be defined and remain a solution.



Example

Find the interval of existence for the solution to the IVP

$$y' = y^2 \quad \text{with } y(0) = 1.$$

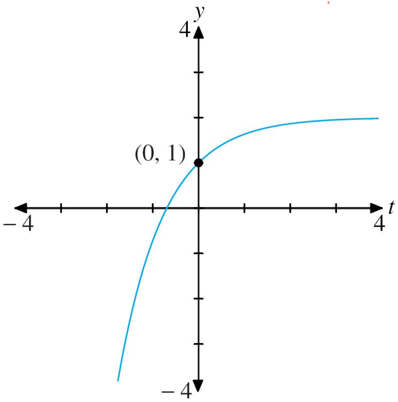
$y(t) = \dfrac{1}{c-t}$, $y(0) = 1$

$y(0) = \dfrac{1}{c-0} = 1 \Rightarrow c = 1$

I.of.E. $= (-\infty, 1)$

Example

$y' = (2-y) \Leftrightarrow \int \frac{dy}{2-y} = \int 1 \cdot dt \Rightarrow -\ln(2-y) = t + C \Leftrightarrow y(t) = 2 - Ce^{-t}$



Example

- Show that $y(t) = 2 - Ce^{-t}$ is a solution of

$$y' = (2-y) \cdot 1$$

for any constant C.

- Find the solution that satisfies the initial condition $y(0) = 1$.
- What is the interval of existence of this solution?

$y(0) = 2 - C \cdot e^{-0} = 1 \Rightarrow C = 1$

I. of E. $= (-\infty, \infty)$

Geometric Meaning of ODE

Geometric Meaning of ODE: Solution Curve and Slopes

Let $y(t)$ be a solution of the ODE

$$y' = f(t, y).$$

The graph of the solution $y(t)$ is called a solution curve. For any point (t_0, y_0) on the solution curve, $y(t_0) = y_0$ and the differential equation says that

$$y'(t_0) = f(t_0, y(t_0));$$

the LHS is the slope of the solution curve, and the RHS tells you what the slope is at (t_0, y_0).

Direction Field

Direction Field for $y' = f(t, y)$

Draw a line segment with slope $f(t_i, y_j)$ attached to every grid point (t_i, y_j) in a rectangle R where $f(t, y)$ is defined

$$R = \{\,(t, y)\,|\, a \le t \le b \text{ and } c \le y \le d\,\}.$$

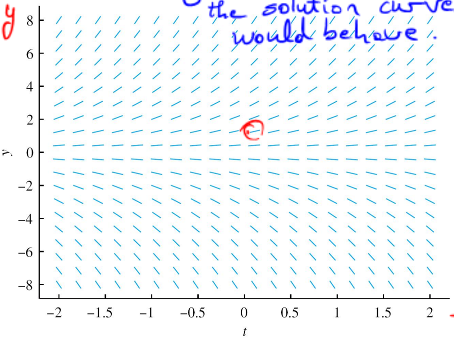
The result is called a $\boxed{\text{direction field.}}$ ← direction fields give an idea of how the solution curves would behave.

$y(0) = 1$

MATLAB:`dfield6` generated the direction field for equation

$$y' = y$$

↳ $y(t) = C \cdot e^t$



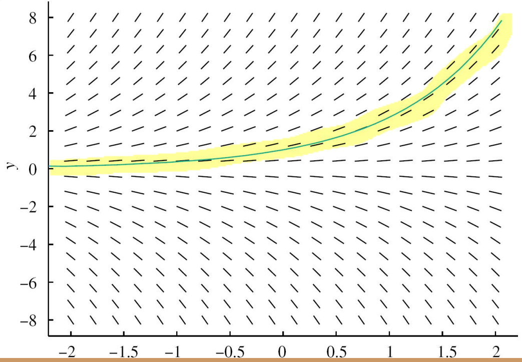
Geometric interpretation of Solutions

Direction field provides information about qualitative form of solution curves.

Finding a solution to the differential equation is equivalent to the geometric problem of finding a curve in ty-plane that is tangent to the direction field at every point.

MATLAB generated the solution curve of

$$y' = y, \quad y(0) = 1$$



Numerical Solution of IVP: Euler's Method

Euler's Method of the Solution of IVP $y' = f(t, y)$, $y(t_0) = y_0$

1) Plot the point $P_0(t_0, y_0)$.
2) Move a prescribed distance along a line with slope $f(t_0, y_0)$ to the point $P_1(t_1, y_1)$.
3) Continue in this manner to produce an approximate solution curve of the IVP.

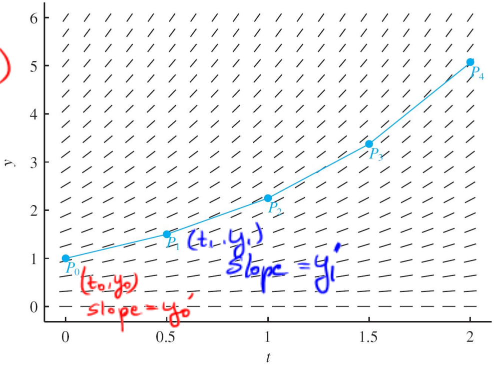
(t_0, y_0) & slope y_0'

$\Rightarrow \boxed{y_1} - y_0 = y_0'(t_1 - t_0)$

MATLAB generated an approximate solution curve of

$$y' = y, \quad y(0) = 1$$

$y_1 = y_0 + y_0'(t_1 - t_0)$



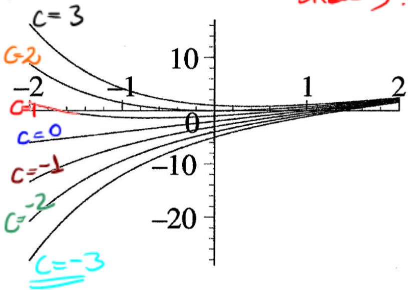
Exercise 2.4

Example (Exercise 2.4)

1). Show that the given solution is a general solution of the differential equation

$$y' + y = 2t, \quad \overbrace{y(t) = 2t - 2 + Ce^{-t}}^{\text{Straightforward to check!}}, \quad C = -3, -2, \cdots, 3$$

Everyone think: (Look at next slide).



2) Use a computer or calculator to sketch members of the family of solutions for the given values of the arbitrary constant.

3) Experiment with different intervals for t until you have a plot that shows what you consider to be the most important behavior of the family.

$y(t) = 2t - 2 + C e^{-t}$

$C=0 \xrightarrow{\text{call}} y_0(t) = 2t - 2 \Leftrightarrow$ line

$C=1 \xrightarrow{\text{call}}$ • $y_1(t) = 2t - 2 + e^{-t}$

Note that as $t \to +\infty$

$y_1(t) \longrightarrow \underline{\underline{2t-2}}$

R.H.S .

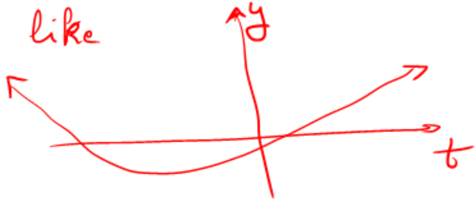
• But as $t \to -\infty$

$y_1(t)$ behaves more like e^{-t} since e^{-t} increases faster than $\|2t-2\|$.

Therefore as $t \to -\infty$

$$y_1(t) = e^{-t} + 2t - 2$$

will look like



We discuss similarly for the rest.

Keep in mind: All these solution curves have nothing in common, they don't touch, they don't intersect.

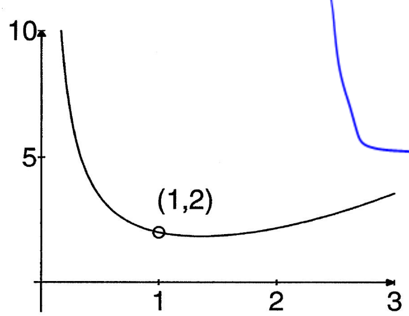
Exercise 2.13

Example (Exercise 2.13)

Use the given general solution to find a solution of the differential equation having the given initial condition. Sketch the solution, the initial condition, and discuss the solutions interval of existence.

check

$$ty' + y = t^2, \quad \overbrace{y(t) = (1/3)t^2 + C/t}^{\text{Check}}, \quad y(1) = 2 \Rightarrow C = \frac{5}{3}$$



$y(t) = (1/3)t^2 + 5/(3t)$.
The interval of existence is $(0, \infty)$.

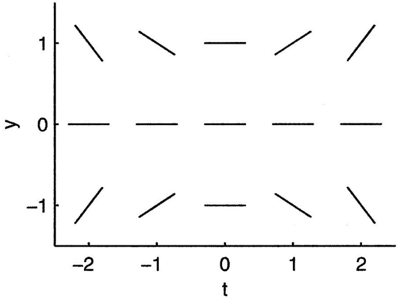
$$y(t) = \frac{1}{3t^2} + \frac{C}{t}$$

is continuous for $t \neq 0$

Since $y(1) = 2$

$\Rightarrow$ Int. E. $= (0, \infty)$.

Exercise 2.19



Example (Exercise 2.19)

Plot the direction field for the differential equation by hand

$$y' = t\tan(y/2).$$

Do this by drawing short lines of the appropriate slope centered at each of the integer valued coordinates (t, y), where $-2 \leq t \leq 2$ and $-1 \leq y \leq 1$

$t=0$, $-1 \leq y \leq 1 \Rightarrow y'=0$

$-2 \leq t \leq 2$, $y=0 \Rightarrow y'=0$

and so the rest.