Math 3331 Differential Equations

2.2 Solutions to Separable Equations

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2.2 Solutions to Separable Equations

- General Method: Separation of Variables
  - Separable Equation
  - Exponential Equation $\rightarrow$ $y' = r \cdot y \Rightarrow y(t) = C \cdot e^{rt}$
  - General Method
  - Explicit Solution
  - Implicitly Defined Solutions
- Applications
  - Radioactive Decay and Half-life
  - Newton's Law of Cooling
- Worked out Examples from Exercises:
  - Find General Solutions: 1, 3, 5, 9, 11
  - Find Solutions to IVPs and IoEs: 13, 15, 17, 19
  - Application: 26, 33

The General Method: Separation of Variables

Form: $\frac{dy}{dt} = g(t)f(y)$

Implicit Solution:

$$[1/f(y)]dy = g(t)dt$$

$$\int [1/f(y)]dy = \int g(t)dt \quad (*)$$

or $H(y) = G(t) + C$ where

$$H(y) = \int [1/f(y)]dy$$

$$G(t) = \int g(t)dt$$

example

$$y' = t^2 + 3t^2 y$$

$$y' = \underbrace{t^2}_{g(t)} \cdot \underbrace{(1+3y)}_{f(y)}$$

Solve $(*)$ for $y \rightarrow$ explicit solution

Note: $(*)$ may have several solutions. Use IC to choose the right one.

Example

Ex.: $\frac{dy}{dt} = ty^2$ $\Longleftrightarrow$ $\int \frac{1}{y^2}\,dy = \int t\,dt$

$$(1/y^2)dy = t\,dt$$

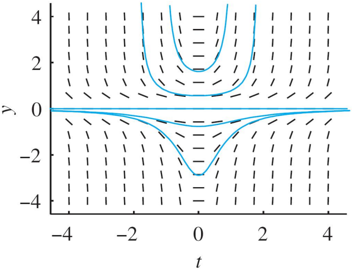
$$\Rightarrow \int (1/y^2)dy = \int t\,dt$$

$$\Rightarrow \boxed{-1/y = t^2/2 + C}$$

$$\Rightarrow y(t) = -1/(t^2/2 + C)$$

$$y(t) = -2/(t^2 + 2C)$$

or $\hookrightarrow$ $y(t) = \dfrac{2}{K - t^2}$, K constant



Example: Exponential Equation

Ex.: Find gen. sol. to $\underbrace{dx/dt = rx}$ recognize

$$\frac{dx}{x} = r\,dt \quad \Rightarrow \quad \ln|x| = rt + C$$

$x' = r \cdot x$

$\boxed{x(t) = Ae^{rt}}$

$$\Rightarrow \quad |x(t)| = e^{rt+C} = e^{C}e^{rt}$$

$$x(t) > 0 \quad \Rightarrow \quad x(t) = e^{C}e^{rt}$$
$$x(t) < 0 \quad \Rightarrow \quad x(t) = -e^{C}e^{rt}$$

Set $A = e^{C}$ if $x > 0$, $A = -e^{C}$ if $x < 0$

$$\Rightarrow \boxed{x(t) = Ae^{rt}}$$

with arbitrary constant A (can be 0)

Initial value: $x(0) = A$

Example: IVP

* This is the example that will help you to get the motion of the with linear air resistance.

Example: general linear equation with constant coefficients

$y' = ry + a$, IC: $y(0) = y_0$ $(r, a, y_0$: arbitrary parameters)

$$[1/(ry+a)]dy = dt \Rightarrow (\ln|ry+a|)/r = t + C \Rightarrow |ry+a| = e^{rt+rC} = e^{rC}e^{rt}$$

$$\Rightarrow ry + a = Ae^{rt} \ (A = \pm e^{rC}) \Rightarrow y(t) = (Ae^{rt} - a)/r$$

Invoke IC: $y(0) = (A-a)/r = y_0 \Rightarrow A = ry_0 + a \Rightarrow y(t) = (y_0 + a/r)e^{rt} - a/r$

$$\int \frac{1}{ry+a}\,dy = \int dt \Rightarrow \frac{1}{r}\ln|ry+a| = t + c$$

$$\Rightarrow \ln|ry+a| = rt + rC \Leftrightarrow |ry+a| = \underbrace{e^{rC}}_{A} \cdot e^{rt}$$

$$\Rightarrow ry + a = Ae^{rt} \Rightarrow \text{next page}$$

$$y(t) = \frac{(Ae^{rt} - a)}{r} \quad , \quad y(0) = y_0$$

$$\Rightarrow y(0) = \frac{A - a}{r} = y_0 \Rightarrow A = ry_0 + a$$

$$\Rightarrow \boxed{y(t) = \left(y_0 + \frac{a}{r}\right)e^{rt} - \frac{a}{r}}$$

Implicitly Defined Solutions

Find sols. of $x' = 2tx/(1+x)$ s.t. $x(0) = 1$, $x(0) = -2$, and $x(0) = 0$.

Answer:

$(1 + 1/x)dx = 2tdt, \quad x \neq 0$

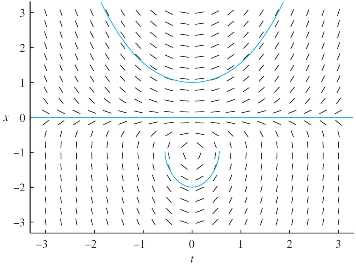
$\Rightarrow x + \ln(|x|) = t^2 + C$

(i) For $x(0) = 1$

$\Rightarrow C = 1$

$x + \ln x - 1 = t^2$

$\Rightarrow x(t)$ implicitly defined.



$x' = \frac{2t\,x}{1+x} = 2t \cdot \left(\frac{x}{1+x}\right)$

$\int \left(\frac{1+x}{x}\right) dx = \int 2t \, dt$

$\boxed{x + \ln|x| = t^2 + C}$

Exercise 2.2.1

Be careful of constant introduced.

Ex. 1: $y' = xy$

$$(1/y)dy = xdx \Rightarrow \ln|y| = x^2/2 + C \Rightarrow |y| = \exp(x^2/2 + C) = e^C e^{x^2/2}$$

$$\Rightarrow y(x) = Ae^{x^2/2}, \; A = e^C \text{ or } A = -e^C$$

Make it as simple as it can get.

Exercise 2.2.3

$y(x) = ?$

Ex. 3: $y' = e^{x-y}$

$$e^y dy = e^x dx \Rightarrow e^y = e^x + C \Rightarrow y(x) = \ln(e^x + C)$$

$\hookrightarrow e^{x-y} = e^x \cdot e^{-y} = \dfrac{e^x}{e^y}$

Exercise 2.2.5

Ex. 5: $y' = y(x+1)$

$(1/y)dy = (x+1)dx \Rightarrow \ln|y| = x^2/2 + x + C \Rightarrow |y| = e^C e^{x+x^2/2} \Rightarrow y(x) = Ae^{x+x^2/2}$

$$\int \frac{dy}{y} = \int (x+1)\,dx$$

$$\Rightarrow y = A \cdot e^{x+\frac{x^2}{2}}$$

Exercise 2.2.9

Ex. 9: $\boxed{x^2y' = y\ln y - y'} \Rightarrow y' = (y\ln y)/(1+x^2)$

$[1/(y\ln y)]dy = [1/(1+x^2)]dx \Rightarrow \ln(\ln y) = \arctan x + C$

$\Rightarrow y(x) = \exp(e^C e^{\arctan x}) = \exp(De^{\arctan x})\ \ (D = e^C)$

$x^2 \cdot y' + y' = y\ln y$

$y'(1+x^2) = y\ln y \Rightarrow y' = \dfrac{y\ln y}{1+x^2}$

Exercise 2.2.11

Ex. 11: $y^3 y' = x + 2y' \Rightarrow y' = x/(y^3 - 2)$

$(y^3 - 2)dy = x\,dx \Rightarrow y^4/4 - 2y = x^2/2 + C \Rightarrow$ implicit sol.: $y^4 - 8y - 2x^2 = D \ (D = 4C)$

Exercise 2.2.13

Ex. 13: $y' = y/x$, IC: $y(1) = -2$

General sol.: $(1/y)dy = (1/x)dx \Rightarrow \ln|y| = \ln|x| + C$

$$\Rightarrow |y| = \exp(C + \ln|x|) = e^C e^{\ln|x|} = e^C|x| \Rightarrow y(x) = Ax \ (A = \pm e^C)$$

Match C to IC: $y(1) = A = -2 \Rightarrow y(x) = -2x$; IoE: $(0, \infty)$

Exercise 2.2.15

Ex. 15: $y' = (\sin x)/y$, IC: $y(\pi/2) = 1$

$y\,dy = \sin x\,dx \;\Rightarrow\; y^2/2 = -\cos x + C \;\Rightarrow\; y = \pm\sqrt{D - 2\cos x}\;\; (D = 2C)$

$y(\pi/2) = 1 > 0 \;\Rightarrow\;$ need '+'-sign $\;\Rightarrow\; y(\pi/2) = \sqrt{D} = 1 \;\Rightarrow\; y(x) = \sqrt{1 - 2\cos x}$

Find IoE: need $\cos x < 1/2 \Rightarrow$ IoE: $(\pi/3, 5\pi/3)$

Why ???

Think !!!

Exercise 2.2.17

Ex. 17: $y' = 1 + y^2$, IC: $y(0) = 1$

$[1/(1+y^2)]dy = dt \ \Rightarrow\ \arctan y = t + C \ \Rightarrow\ y = \tan(t+C) + k\pi \ \ (k:\ \text{integer})$

Since $y(0) = 1 \Rightarrow k = 0 \ \ \Rightarrow\ \ y(t) = \tan(t+C)$

Invoke IC: $y(0) = \tan C = 1 \Rightarrow C = \pi/4 \Rightarrow y(t) = \tan(t + \pi/4)$

For IoE: need $t + \pi/4 > -\pi/2$ and $t + \pi/4 < \pi/2 \Rightarrow$ IoE: $(-3\pi/4, \pi/4)$

Exercise 2.2.19

Ex. 19: $y' = x/y$, IC_1: $y(0) = 1$ and IC_2: $y(0) = -1$

$$y\,dy = x\,dx \;\Rightarrow\; y^2/2 = x^2/2 + C \;\Rightarrow\; y = \pm\sqrt{x^2 + D} \;\; (D = 2C)$$

$$\mathrm{IC}_1\colon y(0) = 1 \;\Rightarrow\; y(0) = +\sqrt{D} = 1 \;\Rightarrow\; y(x) = \sqrt{1 + x^2}$$

$$\mathrm{IC}_2\colon y(0) = -1 \;\Rightarrow\; y(0) = -\sqrt{D} = -1 \;\Rightarrow\; y(x) = -\sqrt{1 + x^2}$$

Radioactive Decay

$N(t)$: $\sharp$ of radioactive atoms

- Model: $dN/dt \sim -N$

$$\Rightarrow \quad dN/dt = -\lambda N$$

- Solution: $N(t) = N_0 e^{-\lambda t}$
- Half–life:

$$N(t)/N(0) = e^{-\lambda t} = 1/2$$

$$\Rightarrow \quad t = (\ln 2)/\lambda \equiv T_{1/2}$$

- Natural log of ratios:

$$\ln[N_0/N(t)] = \lambda t$$

- Use $\lambda = (1/t)\ln[N_0/N(t)]$ to determine λ from measurement
- Use $t = (1/\lambda)\ln[N_0/N(t)]$ to determine time t^* s.t. $N(t^*) = N^*$ for given N^*

Exercise 2.2.25

Ex. 25: After $t = 4\,hrs$, $80\,mg$ of a $100\,mg$ sample of Tritium remain. Determine λ and $T_{1/2}$.

Answer:

$$\lambda = (1/4)\ln[100/80] = 0.056/hrs$$

$$T_{1/2} = (\ln 2)/0.056 = 12.43\,hrs$$

Exercise 2.2.26

Ex. 26: $T_{1/2} = 6\,hrs$ for Technitium 99m. What remains after 9 hrs if $N_0 = 10\,g$?

Answer:

$\lambda = (\ln 2)/6 = 0.116/hr$

$\Rightarrow N(9) = 10e^{-0.116\times 9} = 3.54\,g$

Newton's Law of Cooling

$T(t)$: temperature of objet

A: surrounding temperature

- Model: $dT/dt \sim A - T$

$$\Rightarrow \quad dT/dt = k(A - T)$$

- Solution (see Example p.5):

$$T(t) = A + e^{-kt}(T_0 - A)$$

- $(T - A)/(T_0 - A) = e^{-kt} \Rightarrow$

$$kt = \ln[(T_0 - A)/(T(t) - A)]$$

$k = (1/t)\ln[(T_0 - A)/(T(t) - A)]$
$\rightarrow$ determine k

$t = (1/k)\ln[(T_0 - A)/(T(t) - A)]$
$\rightarrow$ determine t

Exercise 2.2.33

Ex. 33: Dead body found at $t = 0$ (midnight). Temperature $31^0 C$.

1 hr later ($1\,am$): Temperature $29^0 C$

Surrounding temperature: $A = 21^0 C$

Question: When did death (murder) occur?

Answer: $t = 1\,hr$, $T_0 = 31$, $T(1) = 29$

$$\Rightarrow k = (1/1)\ln[(31-21)/(29-21)] = 0.223/hr$$

Determine time at which $T = 37$:

$$t = (1/k)\ln[(31-21)/(37-21)] = -2.11\,hrs = -2\,hrs\,7\,min$$

$\Rightarrow$ Death occured at $9:53\,pm$