Math 3331 Differential Equations

2.3 Models of Motion

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2.3 Models of Motion

- Motion of a Ball Near Surface of the Earth
  - Without Air Resistance
  - With Air Resistance
    - Linear Model
    - Quadratic Model

Motion of a Ball Near Surface of the Earth

$\boxed{a = v' = x''}$

Chapter 1

- **Gravity Force:** $\mathrm{F_g} = -\mathrm{mg} = m \cdot \frac{d^2x}{dt^2} \Rightarrow \boxed{\frac{d^2x}{dt^2} = -g}$
  - $g = 32\,ft/s^2 = 9.8\,m/s^2$
- **Air Resistance:** $\mathrm{F_{air}} = \mathrm{R(v)}$ acts opposite of motion
  - **Linear Model:**

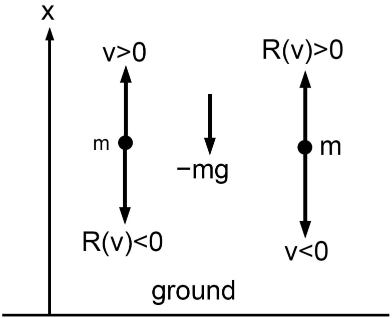
    $m \cdot x'' = -mg + R(v)$

    $$R(v) = -kv$$

    - $[k]$ = mass/time
    - valid for small velocities
  - **Quadratic Model:**

    $$R(v) = -k|v|v$$

    - $[k]$ = mass/length
    - valid for larger velocities
- We treat only the linear model.



Solution of the Motion without Air Resistance

$$v' = -g, \quad x' = v$$

$$\Rightarrow \quad v(t) = -gt + v_0$$

$$x(t) = -gt^2/2 + v_0 t + x_0$$

$$\Rightarrow \quad t = (v_0 - v)/g$$

$$x = (v_0^2 - v^2)/(2g) + x_0$$

$$\Rightarrow \quad v^2 = v_0^2 + 2g(x_0 - x)$$

Max Height (if $v_0 > 0$):

$$v = 0 \quad \Rightarrow \quad t_{max} = v_0/g$$

$$x_{max} = x_0 + v_0^2/(2g)$$

Ground Hit: $x = 0 \Rightarrow$

$$v_g = -\sqrt{v_0^2 + 2gx_0} \text{ (impact velocity)}$$

$$t_g = \left(v_0 + \sqrt{v_0^2 + 2gx_0}\right)/g$$

$x'' = -g$, $x'(0) = v_0$, $\boxed{x(0) = x_0}$

$\hookrightarrow x'(t) = -gt + c$

$\boxed{x'(t) = -gt + v_0} = \underline{v(t)}$

$x(t) = -\frac{gt^2}{2} + v_0 t + C \Rightarrow \boxed{x(t) = gt^2/2 + v_0 t + x_0}$

Example 1

Example 1:

Ascending balloon, velocity $15\,m/s$. At height $100\,m$ package is dropped. When does package reach ground?

$$g = 9.8\,m/s^2$$

Initial Values:

$$x_0 = 100\,m, \;\; v_0 = 15\,m/s$$

$$\Rightarrow \begin{aligned} t_{max} &= 1.5\,s \\ x_{max} &= 111.5\,m \\ v_g &= 46.7\,m/s \\ t_g &= 6.3\,s \end{aligned}$$

Solution of the Motion with Air Resistance

$-mg + R(v)$

$x'' = \boxed{v' = -g - (k/m)v}$ ← separable

$R(v) = -kv$, $k > 0$

In CN 2.2, Example p.5, we showed:

$$
\begin{aligned}
& y' = ry + a, \quad y(0) = y_0 \\
\Rightarrow \quad & y(t) = (y_0 + a/r)e^{rt} - a/r
\end{aligned}
$$

$\frac{v'}{-g - \frac{k}{m} v} = 1$

Here: $y = v$, $a = -g$, $r = -k/m$

$$\Rightarrow \quad v(t) = (v_0 + gm/k)e^{-kt/m} - gm/k$$

look at next slide for details.

Integrate this to find x:

$$
\begin{aligned}
x(t) &= \int_0^t v(t')dt' + x_0 \\
&= \frac{m}{k}(v_0 + gm/k)(1 - e^{-kt/m}) \\
&\quad -(gm/k)t + x_0
\end{aligned}
$$

Terminal Velocity:

$$v_{term} \equiv \lim_{t \to \infty} v(t) = -gm/k$$

$F = -mg - rv$ (object is moving, air resistance involved)

$R = -r \cdot v$ opposite of motion

$$m \cdot \frac{dv}{dt} = -mg - rv$$

$$v' = -g - \frac{r}{m} v = (-1)\left(g + \frac{r}{m} v\right)$$

$$\Rightarrow \frac{v'}{+g + \frac{r}{m} v} = -1 \Rightarrow \int \frac{dv}{g + \frac{r}{m} v} = \int -dt$$

$$\Rightarrow \frac{m}{r} \ln\left|g + \frac{r}{m} v\right| = -t + C_1$$

$$\Rightarrow \ln\left|g + \frac{r}{m} v\right| = \frac{-rt}{m} + C_2$$

$$\Rightarrow \left|g + \frac{r}{m} v\right| = e^{\frac{-rt}{m} + C_2} = e^{C_2} \cdot e^{-\frac{rt}{m}}, \quad A = \pm e^{C_2}$$

$$\Rightarrow \boxed{g + \frac{r}{m} v = A \cdot e^{-rt/m}}$$ Solve for $v \Rightarrow$

$$\frac{r}{m}v = A \cdot e^{-rt/m} - g \Rightarrow v = \underbrace{A \cdot \frac{m}{r}}_{\text{constant}} e^{-rt/m} - \frac{mg}{r}$$

$$\Rightarrow v = B \cdot e^{-rt/m} - \frac{mg}{r}$$

↳ terminal velocity $\underbrace{V_{term} = -\frac{mg}{r}}_{\text{Stability}}$

$$\underbrace{v = \frac{dx}{dt} = B \cdot e^{-rt/m} - \frac{mg}{r}}$$

$$\Rightarrow \int dx = \int \left(B e^{-rt/m} - \frac{mg}{r} \right) dt$$

$$\Rightarrow x(t) = \frac{B \cdot e^{-rt/m}}{-r/m} - \frac{mgt}{r} + C$$

$$x(t) = -\frac{mB}{r} e^{-rt/m} - \frac{mgt}{r} + C.$$

Example 2

Example 2: (see text, Example 3.8)
$m = 2\,kg,\ k = 4\,kg/m\ (g = 9.8\,m/s^2)$
Initial values: $x_0 = 250\,m,\ v_0 = 0$

Question:
Time of ground hit? Impact velocity?

Answer:
Ground hit $\rightarrow$ equation for $t = t_g$:

$$\begin{aligned} 0 &= g(m/k)^2(1 - e^{-kt/m}) - (gm/k)t + x_0 \\ &= 2.45(1 - e^{-2t}) - 4.9t + 250 \\ &= 252.45 - 2.45e^{-2t} - 4.9t \end{aligned}$$

Equation solver $\rightarrow t_g \approx 51.52\,s$

Impact velocity:

$$v_g = v(t_g) = 4.9(e^{-2t_g} - 1) \approx -4,9\,m/s$$

Without air resistance:

$$\begin{aligned} t_g &= \sqrt{2x_0/g} \approx 7.14\,s \\ v_g &= -\sqrt{2gx_0} \approx -44.3\,m/s \end{aligned}$$

Graphs for Example 2