Math 3331 Differential Equations

2.4 Linear Equations

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2.4 Linear Equations

- Homogeneous Equations
- Nonhomogeneous Equations
  - Integrating Factor
  - Variation of Parameter
- Worked out Examples from Exercises:
  - Find General Solutions: 3, 5, 7, 11, 12, 33, 35
  - Find Solutions to IVPs and IoEs: 15, 17, 19, 21, 37, 39

Linear Equations: General Form

General Form:

$$x' = a(t)x + f(t) \qquad (1)$$

If $f(t) = 0$, (1) is called **homogeneous**:

$$x' = a(t)x$$

If $f(t) \neq 0$, (1) is called **nonhomogeneous**

Examples of Linear Equations

Examples of linear equations:

$x' = \sin(t)x$	homogenous, $a(t) = \sin t$
$x' = x/t$	homogenous, $a(t) = 1/t$
$y' = e^t y + \cos t$	nonhomogeneous, $a(t) = e^t$, $f(t) = \cos t$
$x' = 3tx + t^2$	nonhomogeneous, $a(t) = 3t$, $f(t) = t^2$

Examples of Non-Linear Equations

Examples of non-linear equations:

$$
\begin{aligned}
x' &= t \sin x \\
y' &= 1/y \\
y' &= 1 - y^2 \\
u' &= e^{-u} + \cos x
\end{aligned}
$$

Homogeneous Equations

HE: $x' = a(t)x \qquad (2)$

$$\begin{aligned}
\Rightarrow \int \frac{dx}{x} &= \int a(t)dt \\
\Rightarrow \ln|x| &= \int a(t)dt + D \\
\Rightarrow |x(t)| &= \exp\left(D + \int a(t)dt\right) \\
&= e^D \exp\left(\int a(t)dt\right)
\end{aligned}$$

$\Rightarrow$ **General Solution**:

$$x(t) = C\exp\left(\int a(t)dt\right) \qquad (3)$$

where $C = \pm e^D$ (any value)

Equivalent form of gen. sol.:

$$x(t) = x_0 \exp\left(\int_{t_0}^{t} a(t')dt'\right) \qquad (4)$$

where $x_0 = x(t_0)$.

Example

Ex.: $x' = \sin(t)x \quad (a(t) = \sin t)$

$$\Rightarrow \int a(t)\,dt = \int \sin(t)\,dt = -\cos t$$

$$\Rightarrow x(t) = Ce^{-\cos t}$$

Solution:

$x' = \sin t \cdot x$

integrating factor $\int \sin t\,dt = -\cos t$

$\Rightarrow x(t) = C \cdot e^{-\cos t}$

Example

Ex.: $x' = x/t \quad (a(t) = 1/t)$

$$\Rightarrow \int a(t)\,dt = \int \frac{dt}{t} = \ln|t|$$

$$\Rightarrow \quad x(t) = Ce^{\ln|t|} = C|t| \quad (t \neq 0)$$

Since either $t > 0$ or $t < 0$:

$$\Rightarrow \quad x(t) = Bt \quad (t \neq 0,\ B = \pm C)$$

Solution $x' = \left(\frac{1}{t}\right) \cdot x \Rightarrow x(t) = C \cdot e^{\int \frac{1}{t} dt} = \pm C \cdot t$

$\int \frac{1}{t} dt = \ln|t| \Rightarrow e^{\ln|t|}$

$x(t) = B \cdot t$

Nonhomogeneous Equations: Integrating Factor

$x'(t) = a(t)x + f(t)$, $f(t) \neq 0$

NHE: $x' - a(t)x = f(t)$ (5)

Multiply by **integrating factor** $u(t)$ (determined below):

$$u(t)x' - u(t)a(t)x = u(t)f(t) \quad (6)$$

If $u(t)$ satisfies

$$u' = -a(t)u \quad (7)$$

then

$$ux' - uax = ux' + u'x = (ux)'$$

$$\Rightarrow \quad (u(t)x)' = u(t)f(t)$$

$$\Rightarrow \; u(t)x(t) = \int u(t)f(t)dt + C$$

$\Rightarrow$ **General Solution:**

$$x(t) = \frac{1}{u(t)} \int u(t)f(t)dt + C/u(t) \quad (8)$$

where $u(t)$ is a (part.) solution to the HE (7) (cf. (2) and (3)):

$$u(t) = \exp\Big(-\int a(t)dt\Big) \quad (9)$$

Let's generalize: $x' = ax + f$

$\Rightarrow x' + (-a)\cdot x = f$ we need some function u

$u = \int (-a)\,dt$

integrating factor

$\Rightarrow u\cdot x' + (-a)u\cdot x = uf$

$[u\cdot x]' = u\cdot f$

$u(x' - ax) = uf$

$\int d(u\cdot x) = \int u f\,dt$

$\Rightarrow u(t)\cdot x(t) = \int u(t)\cdot f(t)\,dt + C$

$\Rightarrow x(t) = \frac{1}{u(t)} \int u(t) f(t)\,dt + \frac{C}{u(t)}$

To summerize:

$$x' + (-a(t))\,x = f(t)$$

$$\rightarrow u(t) = e^{\int -a(t)\,dt}$$

$$\rightarrow x(t) = \frac{1}{u(t)} \int u(t) f(t)\,dt + \frac{C}{u(t)}$$

Example

Ex.: $x' - x = e^{-t} \ (a(t) = 1, \ f(t) = e^{-t})$

$$\int a(t)\,dt = \int dt = t \ \Rightarrow \ u(t) = e^{-t}$$

$$ux' - ux = e^{-t}x' - e^{-t}x = (e^{-t}x)', \ ue^{-t} = e^{-2t}$$

$$(e^{-t}x)' = e^{-2t} \ \Rightarrow \ e^{-t}x = \int e^{-2t}dt$$

$$e^{-t}x = -e^{-2t}/2 + C \ \Rightarrow \ x(t) = -e^{-t}/2 + Ce^{t}$$

$$x' - x = e^{-t}$$

$$\Rightarrow x' + (-1)x = e^{-t}$$

$$\rightarrow u(t) = e^{\int -1\,dt} = e^{-t}$$

$$x(t) = \frac{1}{e^{-t}} \int e^{-t} \cdot e^{-t} + \frac{C}{e^{-t}}$$

$$= \frac{1}{2} e^{-t} + C e^{t}$$

$$\Rightarrow e^{-t}(x' - x) = e^{-t} \cdot e^{-t}$$

$$\int [e^{-t} \cdot x]' = \int e^{-2t}\,dt \Rightarrow e^{-t} x = -\frac{1}{2} e^{-2t} + C$$

$$x(t) = -\frac{1}{2} e^{-t} + C e^{t}$$

Nonhomogeneous Equations: Variation of Parameter

$$\text{Set } x_h(t) = 1/u(t) \quad (10)$$
$$= \exp\Big(\int a(t)dt\Big)$$

and rewrite (8) as

$$x(t) = Cx_h(t) + x_p(t) \quad (11)$$

where

$$x_p(t) = x_h(t)v(t) \quad (12)$$

with

$$v(t) = \int \frac{f(t)}{x_h(t)}\, dt \quad (13)$$

Eq (12) with (13) is called **Variation of Parameter Formula**

Terms in Gen. Sol. (11):

$Cx_h(t)$: Gen. Sol. of HE (2)
$x_p(t)$: Part. Sol. of NHE (5)

$x'(t) = a(t)\, x + f(t)$ $\Rightarrow$ $x' + (-a(t))\, x = f(t)$

$\hookrightarrow$ homogenous part

$x'(t) = a(t)\, x$

(*) $x_{h(t)} = C \cdot e^{\int a(t)\, dt}$

$u(t) = e^{\int -a(t)\, dt}$

$= e^{-\int a(t)\, dt}$

(**) $= \dfrac{1}{e^{\int a(t)\, dt}}$

$\Rightarrow x_h(t) = \dfrac{1}{u(t)}$ (★) $\Rightarrow$

$\Rightarrow x_p(t) = \dfrac{1}{u(t)} \int u(t) f(t)\, dt$ ★★

$x_h(t) \cdot \int \dfrac{f}{x_h}\, dt$

the general solution is a combination of ★ and ★★

$$\Rightarrow x(t) = x_p(t) + x_h(t)$$

Example

Ex.: $x' = x\tan t + \sin t$

HE: $x' = x\tan t \Rightarrow x_h(t) = \exp\Big(\int \tan t\, dt\Big)$

$\Rightarrow x_h(t) = \exp(\ln(1/\cos t)) = 1/\cos t$

$$\begin{aligned} v(t) &= \int [f(t)/x_h(t)]\, dt \\ &= \int \sin t \cos t\, dt = -\cos^2 t/2 \end{aligned}$$

$$\begin{aligned} x_p(t) &= x_h(t)v(t) = (1/\cos t)(-\cos^2 t/2) \\ &= -\cos t/2 \end{aligned}$$

$\Rightarrow$ Gen. Sol.: $x(t) = -\cos t/2 + C/\cos t$

Example

Example: $y' - ry = f(t)$; use variation of parameter

$$y_h(t) = \exp(\int r\,dt) = e^{rt}, \quad v(t) = \int [f(t)/y_h(t)]dt = \int e^{-rt} f(t)\,dt$$

$$\Rightarrow \text{ gen. sol.: } y(t) = e^{rt}(\int e^{-rt} f(t)\,dt + C)$$

If $f(t) = a = \text{const} \Rightarrow \int e^{-rt} f(t)\,dt = a\int e^{-rt} dt = -(a/r)e^{-rt} \Rightarrow y(t) = Ce^{rt} - a/r$

Exercise 2.4.3

Solve using variation of parameters Method:

$\Rightarrow$ H. form of equation : $y' = \frac{-2}{x} y \Rightarrow y_H(t) = e^{\int -\frac{2}{x} dx} = \frac{1}{x^2}$

Ex. 3: $y' + (2/x)y = (\cos x)/x^2$; use integrating factor

$$a(x) = -2/x \;\Rightarrow\; u(x) = \exp(-\int a(x)dx) = \exp(2\ln x) = \exp(\ln x^2) = x^2$$

Multiply ODE by $x^2 \Rightarrow x^2y' + 2xy = \cos x \;\Rightarrow\; (x^2y)' = \cos x \;\Rightarrow\; x^2y = \int \cos x\, dx$

$$\Rightarrow\; x^2y = \sin x + C \;\Rightarrow\; y(x) = (\sin x + C)/x^2$$

$\Rightarrow$ Particular Solution:

$y_p(t) = \frac{1}{x^2} \int \frac{\frac{\cos x}{x^2}}{1/x^2} dx = \frac{1}{x^2} \int \cos x\, dx = \frac{\sin x}{x^2}$

$\Rightarrow y(t) = y_p(t) + C y_H(t)$

$$y' + \underbrace{\left(\frac{2}{x}\right)} y = \frac{\cos x}{x^2}$$

$$\int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2 \Rightarrow u(x) = e^{\ln x^2} = x^2$$

$$\Rightarrow y(t) = \frac{1}{x^2} \int \underbrace{x^2}_{u} \cdot \underbrace{\frac{\cos x}{x^2}}_{f} dx + \frac{C}{x^2}$$

$$= \frac{1}{x^2} \sin x + C x^{-2}.$$

Exercise 2.4.5

Ex. 5: $x' - 2x/(t+1) = (t+1)^2$; use variation of parameter

HE: $x' = 2x/(t+1) \Rightarrow x_h(t) = \exp(\int [2/(t+1)]dt) = \exp(\ln[(t+1)^2]) = (t+1)^2$

$v(t) = \int [f(t)/x_h(t)]dt = \int dt = t \Rightarrow$ part. sol.: $x_p(t) = x_h(t)v(t) = (t+1)^2 t$

gen. sol.: $x(t) = x_p(t) + Cx_h(t) = (t+1)^2(t+C)$

Exercise 2.4.7

Ex. 7: $(1+x)y' + y = \cos x \Rightarrow y' + y/(1+x) = (\cos x)/(1+x)$; use int. factor

$$u(x) = \exp(-\int a(x)dx) = \exp(\int [1/(1+x)]\, dx) = 1 + x$$

$$\text{Multiply ODE by } u(x) \;\Rightarrow\; (1+x)y' + y = [(1+x)y]' = \cos x$$

$$\Rightarrow\; (1+x)y = \int \cos x\, dx = \sin x + C \;\Rightarrow\; y(x) = (\sin x + C)/(1+x)$$

Exercise 2.4.11

Ex. 11: $y' = \cos x - y \sec x$; use variation of parameter

$$\text{HE: } y' = -y \sec x \;\Rightarrow\; y_h(x) = \exp(-\int \sec x \, dx)$$

$$\Rightarrow\; y_h(x) = \exp(-\ln(\sec x + \tan x)) = 1/(\sec x + \tan x)$$

$$\Rightarrow\; v(x) = \int [f(x)/y_h(x)] dx = \int (\sec x + \tan x) \cos x \, dx = \int (1 + \sin x) dx = x - \cos x$$

$$\Rightarrow\; y_p(x) = y_h(x) v(x) = (x - \cos x)/(\sec x + \tan x)$$

$$\Rightarrow\; y(x) = y_p(x) + C y_h(x) = (x - \cos x + C)/(\sec x + \tan x)$$

Exercise 2.4.12

Ex. 12: $x' - (n/t)x = e^t t^n$; use integrating factor

$$u(t) = \exp(-\int (n/t)dt) = \exp(-n \ln t) = \exp(\ln t^{-n}) = t^{-n}$$

Multiply ODE by $u(t) \Rightarrow (t^{-n}x)' = e^t \Rightarrow (t^{-n}x) = \int e^t dt = e^t + C$

$$\Rightarrow \quad x(t) = t^n(e^t + C)$$

Exercise 2.4.33

Ex. 33: $ty' + y = 4t^2$

Here one sees directly: $(ty)' = ty' + y = 4t^2 \Rightarrow ty = (4/3)t^3 + C$

$$\Rightarrow \quad y(t) = (4/3)t^2 + C/t$$

Exercise 2.4.35

Ex. 35: $y' + 2xy = 4x$; use variation of parameter

$$y_h(x) = \exp(\int(-2x)dx) = e^{-x^2} \Rightarrow v(x) = \int 4xe^{x^2}dx = 2e^{x^2}$$

$$\Rightarrow \; y_p(x) = y_h(x)v(x) = 2 \Rightarrow y(x) = 2 + Ce^{-x^2}$$

Exercise 2.4.15

Ex. 15: $(x^2+1)y' + 3xy = 6x$, $y(0) = -1$; use integrating factor

normal form: $y' + [3x/(1+x^2)]y = 6x/(1+x^2) \;\Rightarrow\; u(x) = \exp(\int [3x/(1+x^2)]dx)$

$\Rightarrow \; u(x) = \exp((3/2)\ln(1+x^2)) = (1+x^2)^{3/2} \;\Rightarrow\; (uy)' = 6x(1+x^2)^{1/2}$

$\Rightarrow \; uy = \int 6x(1+x^2)^{1/2}dx = 2(1+x^2)^{3/2} + C \;\Rightarrow\; y(x) = 2 + C(1+x^2)^{-3/2}$

Invoke IC: $y(0) = 2 + C = -1 \;\Rightarrow\; C = -3 \;\Rightarrow\; y(x) = 2 - 3(1+x^2)^{-3/2}$

Exercise 2.4.17

Ex. 17: $x' + x\cos t = (1/2)\sin 2t = \sin t \cos t$, $x(0) = 1$; use variation of parameter

$$x_h(t) = \exp(-\int \cos t\, dt) = e^{-\sin t} \Rightarrow v(t) = \int \sin t \cos t e^{\sin t} dt = e^{\sin t}(\sin t - 1)$$

$$\Rightarrow x_p(t) = \sin t - 1 \Rightarrow \text{gen. sol.: } x(t) = \sin t - 1 + Ce^{-\sin t}$$

$$\text{Invoke IC: } x(0) = -1 + C = 1 \Rightarrow C = 2 \Rightarrow x(t) = \sin t - 1 + 2e^{-\sin t}$$

Exercise 2.4.19

Ex. 19: $(2x+3)y' = y + (2x+3)^{1/2}$, $y(-1) = 0$; use integrating factor

normal form: $y' = y/(2x+3) + (2x+3)^{-1/2} \Rightarrow u(x) = \exp(\int [-1/(2x+3)]dx)$

$$\Rightarrow u(x) = \exp(-(1/2)\ln(2x+3)) = (2x+3)^{-1/2} \Rightarrow (uy)' = (2x+3)^{-1}$$

$$\Rightarrow uy = \int (2x+3)^{-1} dx = (1/2)\ln(2x+3) + C$$

$$\Rightarrow y(x) = (1/2)(2x+3)^{1/2}(\ln(2x+3) + C), \text{ IC } \Rightarrow y(-1) = C/2 = 0$$

$$\Rightarrow y(x) = (1/2)(2x+3)^{1/2}\ln(2x+3), \text{ IoE: } (-3/2, \infty)$$

Exercise 2.4.21

Ex. 21: $(1+t)x' + x = \cos t,\ x(-\pi/2) = 0$

One sees directly: $[(1+t)x]' = (1+t)x' + x = \cos t \Rightarrow (1+t)x = \int \cos t\, dt$

$$\Rightarrow (1+t)x = \sin t + C \ \Rightarrow\ x(t) = (\sin t + C)/(1+t)$$

$x(-\pi/2) = (-1+C)/(1-\pi/2) = 0 \ \Rightarrow\ C = 1 \ \Rightarrow\ x(t) = (1+\sin t)/(1+t)$, IoE: $(-\infty, -1)$

Exercise 2.4.37

Ex. 37: $y' + y/2 = t$, $y(0) = 1$;

From Example p.7 $(r = -1/2,\ f(t) = t)$: $y(t) = e^{-t/2}(\int e^{t/2} t\, dt + C)$

$$\int e^{t/2} t\, dt = 2(t-2)e^{t/2} \ \Rightarrow\ y(t) = 2(t-2) + Ce^{-t/2}$$

IC: $y(0) = -4 + C = 1 \ \Rightarrow\ C = 5 \ \Rightarrow\ y(t) = 2(t-2) + 5e^{-t/2}$

Exercise 2.4.39

Ex. 39: $y' + 2xy = 2x^3$, $y(0) = -1$; use variation of parameter

$$y_h(x) = \exp\left(\int (-2x)dx\right) = e^{-x^2} \Rightarrow v(x) = \int 2x^3 e^{x^2} dx = (x^2-1)e^{x^2} \Rightarrow y_p(x) = x^2-1$$

$$\Rightarrow y(x) = x^2 - 1 + Ce^{-x^2}, \text{ IC: } y(0) = C - 1 = -1 \Rightarrow C = 0 \Rightarrow y(x) = x^2 - 1$$