Math 3331 Differential Equations

2.7 Existence and Uniqueness of Solutions

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2.7 Existence and Uniqueness of Solutions

- Existence of Solution
  - Existence for Linear Equation
  - Existence when the Right-hand Side is Discontinuous
- Interval of Existence of a Solution
- Uniqueness of Solution
- Worked out Examples from Exercises:
  - 1, 3, 5, 7

Existence and Uniqueness Theorem

Basic Existence and Uniqueness Theorem (EUT):

Suppose $f(t,x)$ is defined and continuous, and has a continuous partial derivative $\partial f(t,x)/\partial x$ on a rectangle R in the tx−plane. Then, given any initial point (t_0, x_0) in R, the initial value problem

$$x' = f(t,x), \quad x(t_0) = x_0$$

from $\boxed{R = \text{nice domain}}$

has a unique solution $x(t)$ defined in an interval containing t_0. Furthermore, the solution will be defined at least until the solution leaves R.

→ existence of solutions depends on continuity of $f(t,x)$.

→ uniqueness of solutions depends on "smoothness" of $f(t,x)$, i.e $\frac{\partial f}{\partial x}$ should be continuous.

Example

Ex.: $\boxed{tx' = x + 3t^2} \Rightarrow \underline{x' = x/t + 3t}$, $x(0)=1$

Let's break it down in next page:

- f and $\partial f/\partial x$ are defined and continuous for any (t, x) if $t \neq 0$
- General solution (use Sec. 2.4):

$$x(t) = 3t^2 + Ct \quad , \; x(0) = 1$$

$x(0)=0$

- For any C: $x(0) = 0$, hence $\quad x(0) = 0 \neq 1$
  - no solution for $x(0) = x_0 \neq 0$
  - ∞ solutions for $x(0) = 0$
- Solution for $x(t_0) = x_0$, $t_0 > 0$:

$$3t_0^2 + Ct_0 = x_0 \;\Rightarrow\; C = x_0/t_0 - 3t_0$$

$$\Rightarrow\; x(t) = 3t^2 + (x_0/t_0 - 3t_0)t$$

unique solution with IoE $(0, \infty)$

- EUT applies to any rectangle that is not intersected by the vertical line $t = 0$.

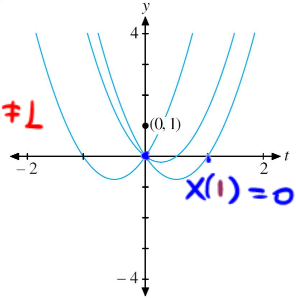


Figure 1 All solutions of (7.2) pass through (0,0).

We have $tx' = x + 3t^2 \Rightarrow$

$\Rightarrow$ Normal Form $x' = \frac{1}{t}x + 3t$ $\leftarrow$ $u(t) = e^{\int -\frac{1}{t}dt} = \frac{1}{t}$ integrating factor

Using one of methods, we get solution

general solution $x(t) = 3t^2 + Ct$

<u>Case I</u>: IVP, $x' = \frac{1}{t}x + 3t$, $x(0) = 1$

equation is linear but it makes no sense at $t=0$, since $\frac{1}{t}$ is undefined.

$\Rightarrow$ So, if we stay away from $t=0$ then every solution is of form

$t \neq 0 \longrightarrow \boxed{x(t) = 3t^2 + C\cdot t}$, for some constant C.

Notice that $x(t) = 3t^2 + Ct$ are defined for all values of t, including 0, and $x(0) = 0$.

They are solution to $tx' = x + 3t^2$ even for $t=0$.

Thus, to conclude, if we want to solve the given initial value problem, we make sure that the initial conditions are included in the "rectangle domain" of $f(t,x)$ = right hand side of a normal form of ODE.

Example: Non-Uniqueness of Solution

$x(1)=3$ sep. $\int \frac{dx}{x^{1/3}} = \int dt$ solve for x

Look on next page for solution

Ex.: $x' = x^{1/3}$

S.o.V.: $\int x^{-1/3}dx = \boxed{(3/2)x^{2/3} = t + D}$

$\Rightarrow \; x_{\pm}(t) = \pm[(2/3)t + C]^{3/2} \quad (C = 2D/3)$

$x(0)=0$

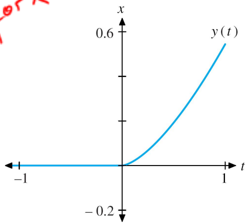
- Let $C = 0 \Rightarrow x_{\pm}(0) = 0$
- Other solution with $x(0) = 0$:

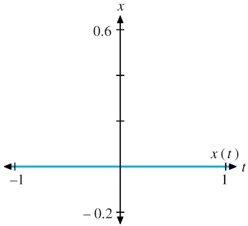
$$\boxed{x(t) = 0}$$

$\Rightarrow$ At least 3 solutions for IC

$$x(0) = 0$$

- EUT doesn't apply to any rectangle that is intersected by the horizontal line $x = 0$





$x' = x^{1/3} \Rightarrow \int \frac{dx}{x^{1/3}} = \int dt \Rightarrow \frac{3}{2} x^{2/3} = t + D$

solve for x

$\Rightarrow x^{2/3} = \frac{2}{3}(t+D) = \frac{2}{3}t + C$, $C = \frac{2}{3}D$

Being a square, we get two options:

$$x(t) = \pm \left(\frac{2}{3}t + C\right)^{3/2}$$

- if $x(0) = 0$, then $C = 0$.
  Then either $x(t) = \left(\frac{2}{3}t\right)^{3/2}$, or $x(t) = -\left(\frac{2}{3}t\right)^{3/2}$

- if $x(0) = 0$, also $x(t) = 0 \leftarrow$ constant function is another solution.

• if $x(0)=0$, another solution can be

$$y(t) = \begin{cases} \left(\frac{2}{3}t\right)^{3/2}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and you can built other solutions.

So, the system is not nice, you get a lot of solutions for a initial value problem.

This happens because, the given initial value is a point where $f(t,x)$ has not a continuous derivative!!!

$$\frac{\partial f}{\partial x}\left(x^{1/3}\right) = \frac{1}{3} \cdot x^{-2/3}, \quad x \neq 0$$

Interval of Existence

Interval of Existence:

Largest interval in which a solution of a first order ODE can be defined.

Example

Existence, Uniqueness Theorem

$f(t,x) = -x^2$

1. $f(t,x)$ continuous

2. $\frac{\partial f}{\partial x} = -2x$ continuous

Ex.: $x' = -x^2$, $x(0) = x_0$

S.o.V.: $-\int (1/x^2)dx = 1/x = t + C$

$$\Rightarrow \; x = 1/(t+C)$$

$$x(0) = 1/C = x_0 \Rightarrow C = 1/x_0$$

$$\Rightarrow \; x(t) = x_0/(1 + x_0 t)$$

If $\left.\begin{array}{l} x_0 > 0 \\ x_0 < 0 \end{array}\right\} \Rightarrow$ IoE: $\left\{\begin{array}{l} (-1/x_0, \infty) \\ (-\infty, -1/x_0) \end{array}\right.$

If $x_0 = 0 \Rightarrow \boxed{x(t) = 0}$, IoE: $(-\infty, \infty)$

- $f(t,x) = -x^2$ satisfies hypotheses of EUT in any rectangle

$\Rightarrow$ Unique solution for any x_0

- $x(t)$ leaves any rectangle in finite time

$\Rightarrow$ Solution is not defined for *all* reals if $x_0 \neq 0$

Details explained in the next slides.

$\Longrightarrow$ solutions exist and are unique!

$x' = -x^2$, $x(0) = x_0$

Note that $f(t,x) = -x^2$ is continuous and $\frac{\partial f}{\partial x} = -2x$ is continuous. Therefore, this equation has a solution and it is going to be unique for every value $x(0) = x_0$.

$$x' = -x^2 \Rightarrow \int \frac{-dx}{x^2} = \int dt \Rightarrow \frac{1}{x} = t + C$$

$$\Rightarrow x(t) = \frac{1}{t+C}$$, all possible curves.

Depending on initial value $x(0) = x_0$, we get

$$x(0) = \frac{1}{0+C} = x_0 \Rightarrow C = \frac{1}{x_0}$$

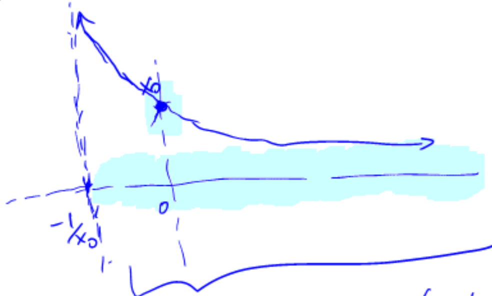
Therefore, depending on x_0, $x(t) = \dfrac{1}{t + \frac{1}{x_0}}$.

i.e. Unique solution $x(t) = \dfrac{x_0}{x_0 t + 1}$.

Case I: If $x(0) = x_0 > 0$, then

$x(t) = \dfrac{1}{t + \frac{1}{x_0}}$ has interval of existence $\left(-\frac{1}{x_0}, \infty\right)$

$t \neq -\frac{1}{x_0}$

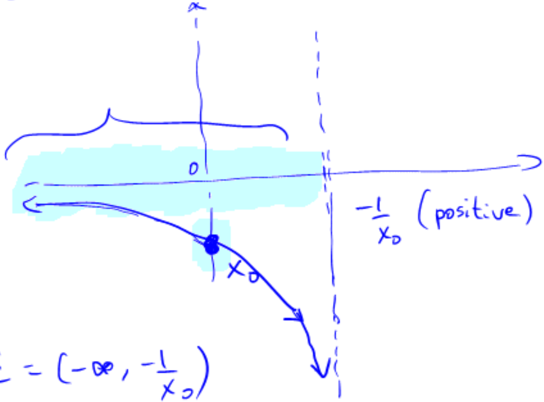


I. of. E $= \left(-\frac{1}{x_0}, \infty\right)$

Case II. $x(0) = x_0 < 0$

$x(t) = \dfrac{1}{t + \frac{1}{x_0}}$ has interval of existence $\left(-\infty, -\frac{1}{x_0}\right)$

$t \neq -\frac{1}{x_0}$



I. of. E $= \left(-\infty, -\frac{1}{x_0}\right)$

Case III. $x(0) = x_0 = 0$

Note this value cannot be calculated from $x(t) = \frac{1}{t+C}$ since this curve never reaches '0'.

- In such situations, look for constant values $f(t,x)$ can get. Hence, only for $\boxed{x(t) = 0}$, it works.

Interval of existence, $(-\infty, \infty)$.

Existence When the RHS is Discontinuous

Ex.: IVP $y' = -2y + \boxed{f(t)}$, $y(0) = 3$

$$f(t) = \begin{cases} 0 & \text{if} \quad t < 1 \\ 5 & \text{if} \quad t \geq 1 \end{cases}$$

$\rightarrow$ $t < 1$: $\boxed{y' = -2y}$ $\Rightarrow$ $y(t) = 3e^{-2t}$, $y(0)=3$

For $t \to 1$: $y(1) = 3e^{-2}$

Continue solution beyond $t = 1$:

$t \geq 1$: $y' = -2y + 5$, $\underline{y(1) = 3e^{-2}}$

$$\Rightarrow y(t) = 3e^{-2t} + e^{-2t} \int_1^t e^{2t'} 5\, dt'$$

$$y(t) = \boxed{5/2 + (3 - 5e^2/2)e^{-2t}}$$

Combine:

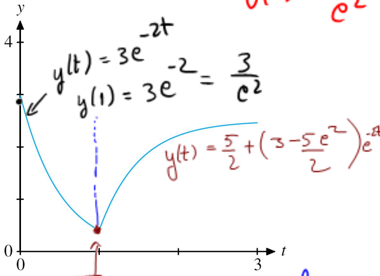
$$y(t) = \begin{cases} 3e^{-2t} & \text{if} \quad t \leq 1 \\ 5/2 + (3 - 5e^2/2)e^{-2t} & \text{if} \quad t \geq 1 \end{cases}$$

- f is discontinuous at $t = 1$, but unique solution exists for all t
- $y'(t)$ is discontinuous at $t = 1$

$\Rightarrow$ I $y' = -2y + 0$, $t < 1$

II $y' = -2y + 5$, $t \geq 1$

$\hookrightarrow y(1) = \frac{3}{e^2}$



$\boxed{y(1)}$ is the initial condition for 2^{nd} part of this eqn.

I. $y' = -2y + 0$, $\quad y(0) = 3$ $\quad , t < 1$

II $y' = -2y + 5$, $\quad ?$ $\quad , t \geq 1$

I: $y' = -2y$, $y(0) = 3$

$\hookrightarrow \; y_1(t) = Ce^{-2t} \implies y_1(t) = 3e^{-2t}$.

$y_1(t) = 3e^{-2t}$, for all $t < 1$

We stop at $t = 1$, by continuity

$$\boxed{y_1(1) = 3e^{-2}}$$

II. $y' = -2y + 5$ begins at $y_2(1) = \frac{3}{e^2}$, $t \geq 1$.

$$\Rightarrow y_2(t) = e^{-2t} \int \frac{5}{e^{-2t}}\, dt$$

$$= e^{-2t}\left[\frac{5}{2}e^{2t} + C\right] = \frac{5}{2} + Ce^{-2t}$$

$$y(1) = \frac{3}{e^2} \Rightarrow \boxed{C = 3 - \frac{5}{2}e^2}$$

$$y_2(t) = \frac{5}{2} + \left(3 - \frac{5}{2}e^2\right)e^{-2t}, \quad t \geq 1$$

$$\Rightarrow \boxed{y(t) = \begin{cases} y_1(t), & t < 1 \\ y_2(t), & t \geq 1 \end{cases} = \begin{cases} 3e^{-2t}, & t < 1 \\ \frac{5}{2} + \left(3 - \frac{5}{2}e^2\right)e^{-2t}, & t \geq 1 \end{cases}}$$

Exercise 2.7.1

$f(t,y)$

Ex. 1: $y' = 4 + y^2$, $y(0) = 1$. Does IVP have a unique solution?

Yes, because $f = 4 + y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere.

Exist: $f(t,y) = 4 + y^2$ continuous

Unique: $\frac{\partial f}{\partial y} = 2y$ continuous

By theorem, the solution would be unique.

Exercise 2.7.3

Ex. 3: $y' = t \tan^{-1}(y)$, $y(0) = 2$. Does IVP have a unique solution?

Yes (as Ex. 1).

existence: $f(t,y) = t \cdot \tan^{-1}(y)$ continuous

uniqueness: $\frac{\partial f}{\partial y} = t \cdot \frac{1}{1+y^2}$ continuous

$\Rightarrow$ The solution is unique for $y(0)=2$.

Exercise 2.7.5

$$f(t,x) = \frac{t}{x+1}$$

Ex. 5: $x' = t/(x+1)$, $x(0) = 0$. Does IVP have a unique solution?

Yes, because f and $\partial f/\partial x = -t/(x+1)^2$ are continuous in any rectangle away from the horizontal line $x = -1$, and $x(0) \neq -1$.

For all $x \neq -1$, $f(t,x)$ is continuous

and $\dfrac{\partial f}{\partial x} = \dfrac{-t}{(x+1)^2}$ is continuous

Hence, solution exists and is unique since $\boxed{x(0) \neq -1}$

Exercise 2.7.7

$y' = \frac{1}{t} y + t \cos t$

Rectangle domain $\underline{t \neq 0}$

Ex. 7: $ty' - y = t^2 \cos t$, $y(0) = -3$.

$t=0$ is "anomaly"

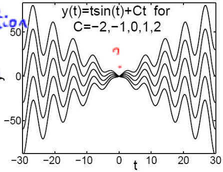
(i) Find general solution and sketch several solutions.
(ii) Show IVP has no solution and explain why this doesn't contradict EUT.

Answer (i): $y' - y/t = t \cos t$, use integrating factor:

$$u(t) = \exp(-\int (1/t)dt) = \exp(-\ln t) = 1/t$$

$$\Rightarrow (y/t)' = \cos t \Rightarrow y/t = \sin t + C \Rightarrow \boxed{y(t) = t \sin t + Ct}$$

general solution



Answer (ii): Since $y(0) = 0$ for any C, there is no solution that satisfies $y(0) = -3$. This doesn't contradict EUT because f is not continuous at $t = 0$.

(ii) We are taking the "anomalie" direction.
$t=0 \Longrightarrow$ solutions not unique for that point. But $y(0) = 0$, hence no solution