Math 3331 Differential Equations

2.8 Dependence of Solutions on Initial Conditions

Blerina Xhabli

Department of Mathematics, University of Houston

`blerina@math.uh.edu`
`math.uh.edu/∼blerina/teaching.html`

2.8 Dependence of Solutions on Initial Conditions

- Continuity with respect to Initial Conditions
- Sensitivity to Initial Conditions

Keep in mind: Solutions to Initial Value problems are "continuous" smooth curves most of the time.

Dependence of Solutions on Initial Conditions

How can we compare the real solution curve to some solution with incorrect initial data but very close to real one!!!

Q1. Continuity of the solution with respect to initial data: Can we ensure that the solution with incorrect initial data is close enough to the real solution that we can use it to predict behavior?

Q2. Sensitivity to initial conditions: Given that we have an error in the initial conditions, just how far from the true solution can the solution be?

There must be an error!
Question is how we minimize it!

Theorem 7.15

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Suppose the function $f(t, x)$ and its partial derivative $\frac{\partial f}{\partial x}$ are both continuous on the rectange R in the tx-plane and let

$x' = f(t,x)$

normal part

$$M = \max_{(t,x)\in R} \left| \frac{\partial f}{\partial x} \right|$$

$\left.\frac{\partial f}{\partial x}\right|_R$ = a lot of values → take max.

Suppose (t_0, x_0) and (t_0, y_0) are in R and that

$$x'(t) = f(t, x(t)), \quad \text{and } x(t_0) = x_0$$
$$y'(t) = f(t, y(t)), \quad \text{and } y(t_0) = y_0$$

At t_0, x_0 and y_0 are very close

Then as long as $(t, x(t))$ and $(t, y(t))$ belong to R, we have

$$\text{error} = |x(t) - y(t)| \le |x_0 - y_0| e^{M|t-t_0|}$$

real pretended solution

Why $|x(t)-y(t)| \le |x_0-y_0|\, e^{M|t-t_0|}$, R-domain

where $M = \max_R \left| \frac{\partial f(t,x)}{\partial x} \right|$

Back to Calculus:

Think of a continuous and differentiable function, say $f(x)$. We want to find the linearization of f at some point "x_1".

$$L_1(x) = f(x_1) + f'(x_1)(x - x_1)$$

Think of another point "x_2" very very close to x_1.

Again we find linearization function:

$$L_2(x) = f(x_2) + f'(x_2)(x-x_2).$$

Being very very close for x_1 and x_2, it means that the error between L_1 and L_2 is very small.

What is this error exactly ???

$$L_1(x) - L_2(x) = f(x_1) - f(x_2) + \underbrace{f'(x_1)(x-x_1)} - \underbrace{f'(x_2)(x-x_2)}$$

x_1 being very close to x_2 implies that these two terms eleminate each other

$$\Rightarrow \quad L_1(x) - L_2(x) \approx \underbrace{f(x_1) - f(x_2)}_{\text{Mean Value Theorem}}$$

$$\cong \underbrace{f'(c)}_{\text{maximizing over the domain}} (x_1 - x_2)$$

$$\Rightarrow$$

$$\left| L_1(x) - L_2(x) \right| \leq \underbrace{\max_{\text{domain}} |f'(x)| \; |x_1 - x_2|}$$

We have two approaches being closed by this error

Back to our problem : derivative of $x(t)$ or $y(t)$

$$|x(t) - y(t)| \le \max|f'(t,x)| \cdot |x_0 - y_0|$$

two solutions

initial solutions

And theorem says

$$|x(t) - y(t)| \le e^{M|t-t_0|} |x_0 - y_0|$$

i.e. roughly speaking $\max|f(t,x)| \le e^{M|t-t_0|}$

How???

We just know $\max_R \left| \frac{\partial f}{\partial x} \right| = \underline{M}$.

$e^{\underline{M}|t-t_0|}$ corresponds to $\max |f(t,x)|$

(roughly speaking)

We'll continue on Monday, 02/08.

Example 2.8.1: Continuity w.r.t. Initial Conditions

Example 2.8.1: Consider $x' = \underbrace{(x-1)\cos t}_{f(t,x)}$. Since $\Rightarrow \frac{\partial f}{\partial x} = \cos t$

$$M = \max_{(t,x)\in R} \left|\frac{\partial f}{\partial x}\right| = \max_{(t,x)\in R} |\cos t| \le 1$$

$|x_0 - y_0| < 0.1$

then

$$|x(t) - y(t)| \le \underbrace{|x_0 - y_0|}_{0.1} e^{|t-t_0|}, \qquad \text{for all } t.$$

black
• x(t) ↔ real solution with x(0)=0

① y(t) – the solution with y(0) = 0.1

② z(t) – the solution with z(0) = –0.1

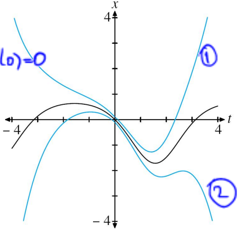


Figure 1 A solution to (8.2) with $|x(0)| \le 0.1$ must lie between the colored curves.

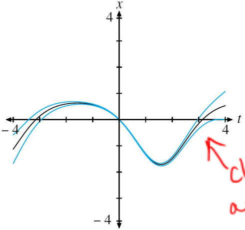


Figure 2 A solution to (8.2) with $|x(0)| \le 0.01$ must lie between the colored curves.

Similarly, here.

closer you get as initial values better approach to the real you see

Example 2.8.6: Sensitivity to Initial Conditions

Example 2.8.6: Consider $x' = \underbrace{x \sin x + t}_{f(t,x)}$.

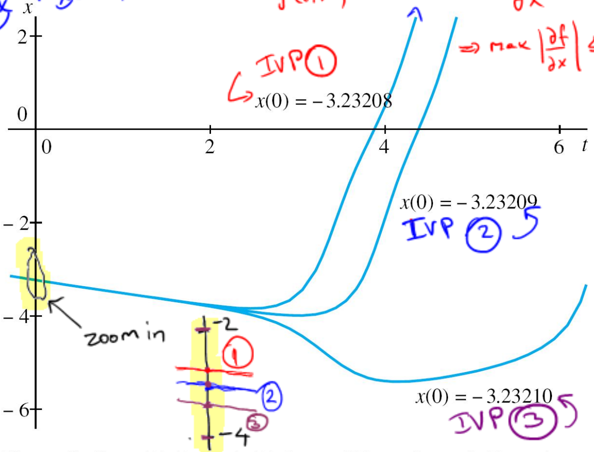
$\frac{\partial f}{\partial x} = \sin x + x \cdot \cos x$

$\left|\frac{\partial f}{\partial x}\right| = |\sin x + x \cos x|$

$\Rightarrow \max \left|\frac{\partial f}{\partial x}\right| \le |\sin x| + |x| \cdot |\cos x|$

$\le 1 + |x| \cdot 1$

$|x|$ depends on R.



Keep in mind solutions do not meet, because they are unique!!!

Figure 3 Sensitivity to initial conditions for solutions to $x' = x \sin(x) + t$.