Math 3331 Differential Equations

2.9 Autonomous Equations and Stability

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2.9 Autonomous Equations and Stability

- Autonomous Equations
- Equilibrium Points and Solutions
- Direction Field and Stability of Equilibrium
  - Example: Falling Object and Terminal Velocity
- Qualitative Analysis
  - Properties of Solutions
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As I said last time, 'til now we've worked with "linear differential eqns" $\longleftrightarrow$ $x' = a(t)\cdot x + b(t)$, "separable equations of form" $\longleftrightarrow$ $x' = h(t)\cdot f(x)$, etc.

In this section, we'll focus on separable equations, whose normal form is "independent" of variable "t".

i.e. $\boxed{x' = f(x)}$. $\longleftarrow$ will be called "autonomous".

Autonomous Equations

Form: $x' = f(x)$

Implicit Solution:

$$\int [1/f(x)]\,dx = \int dt$$

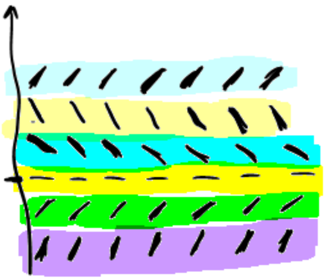
$$\Rightarrow \; G(x) = t + C$$

where $G(x) = \int [1/f(x)]\,dx$ is an antiderivative of $1/f(x)$

Consequence: If $x(t)$ is solution $\Rightarrow x(t+C)$ is solution

$x' = f(x) \rightarrow$ independent of time t.

Direction field of such equations consists of of slopes that do not change horizontally:

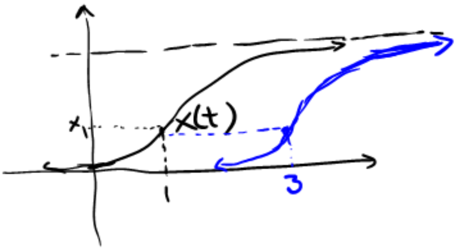


This feature is very important because it shows that the slopes will never change even if you move left or right.

Hence, if $x(t)$ is a solution to $x' = f(x)$.

Then for sure $x_1(t) = x(t+c)$ will be some other solution, c - constant of horizontal shiftment

This is nice, because if you have already a solution $x(t)$ for your equation $x' = f(x)$



then

you can obtain a new solution curve by shifting the known solution left or right.

For example, $y(t) = x(t-2)$ would be another solution to $x' = f(x)$

Never forget, Solutions are unique, they don't touch.

Examples

autonomous

Ex: $x' = \sin(x)$, $y' = \underbrace{y^2 + 1}_{\text{autonomous}}$

are autonomous

$x' = \sin(tx)$, $y' = xy$

are *not* autonomous

Most of 1st order differential equations have special constant solutions: $x(t) = x_0$.

- we find them by doing $f(x) = 0$.

example:

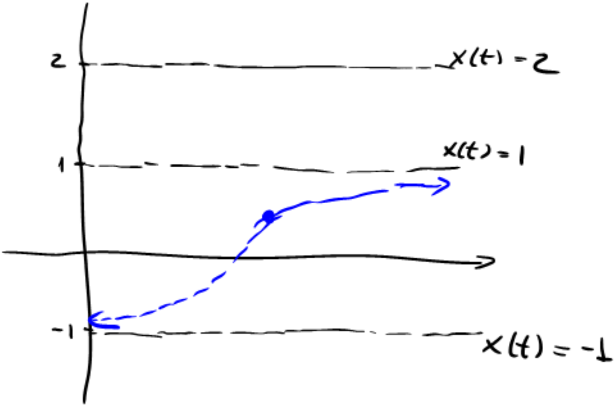
$$x' = \underbrace{(x^2-1)(x-2)}$$

$f(x) = 0 \Rightarrow x = 2, 1, -1$.

Hence, 3 solutions are

$$\boxed{\begin{aligned} x(t) &= 1 \\ x(t) &= -1 \\ x(t) &= 2 \end{aligned}}$$

These solutions are important for behaviour of other solutions.



(I'll show later in other slide why increasing)

Since solutions are unique, if you begin with some initial value between constant function

— then you should stay within those parallel lines. Look at blue curve.

Equilibrium Points and Solutions

"Constant" solutions of $f(x)=0$.

Equilibrium Point $\mathbf{x_0}$:

Solution of $f(x_0) = 0 \Rightarrow$

$x(t) = x_0$ is constant solution

$v(t) = C\cdot e^{-kt/m} - \frac{mg}{k}$

Ex.: $v' = -g - kv/m$

$v(t) \xrightarrow[t\to\infty]{} -\frac{mg}{k}$

$f(v) = 0 \Rightarrow v_{term} = -gm/k$

is equilibrium point

(Falling Object, Air Resistance and Terminal Velocity)

Remember from Section 2.3

$$v' = -g - \underbrace{\frac{kv}{m}}_{f(v)}$$

The solution curves

$$v(t) = Ce^{-kt/m} - \frac{mg}{k}$$

and

$$v(t) \xrightarrow[t \to +\infty]{} -\frac{mg}{k}$$

$$f(v) = -\frac{k}{m} v - g$$

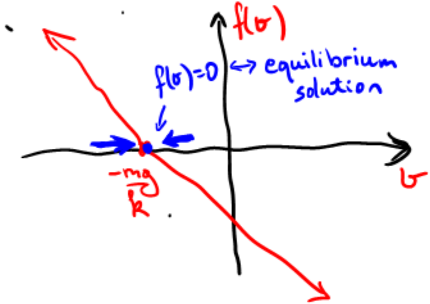
$\hookrightarrow f(v) = 0 \Rightarrow v(t) = \frac{-mg}{k}$

$$v_{terminal} = \frac{-mg}{k}$$

"equilibrium"

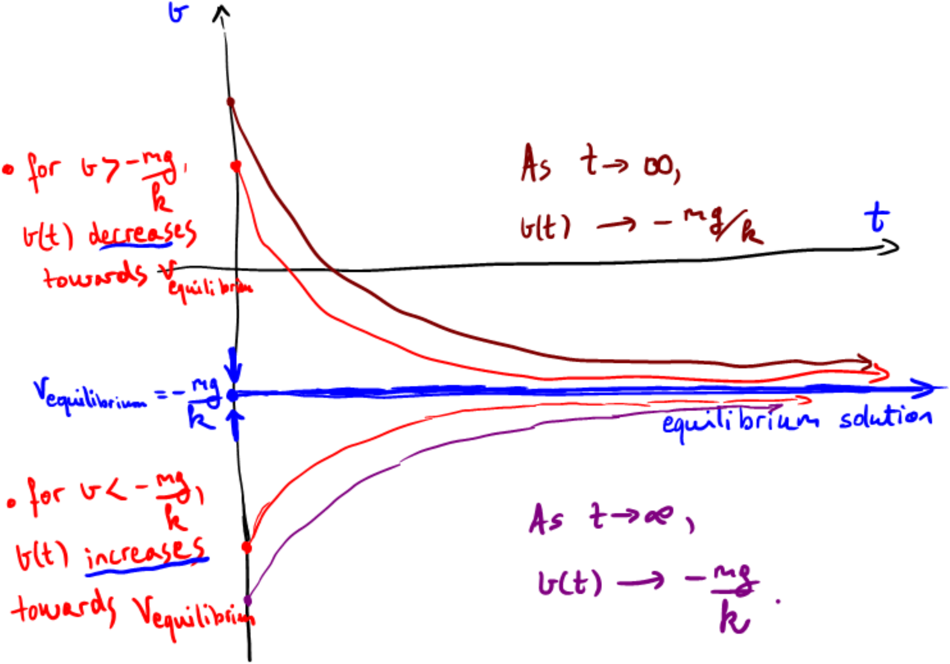
Thus, "constant" solutions to diff. equations are "equilibrium" solutions to the system.

Graph $f(v) = -\frac{k}{m}v - g$



Red line is the derivative graph for solutions $v(t)$.

- if $v < -\frac{mg}{k}$, $f(v) > 0$, hence $v(t)$ increasing towards $-\frac{mg}{k}$.
- if $v > -\frac{mg}{k}$, $f(v) < 0$, hence $v(t)$ decreasing towards $-\frac{mg}{k}$.



v
t
• for $v > -\frac{mg}{k}$,
$v(t)$ decreases
towards $v_{equilibrium}$
As $t \to \infty$,
$v(t) \to -mg/k$
$v_{equilibrium} = -\frac{mg}{k}$
equilibrium solution
• for $v < -\frac{mg}{k}$,
$v(t)$ increases
towards $v_{equilibrium}$
As $t \to \infty$,
$v(t) \to -\frac{mg}{k}$.

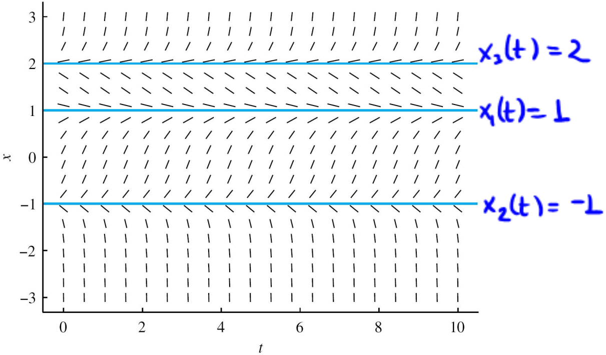
If the solutions $x(t)$ of $x' = f(x)$ on both sides of an equilibrium point x_0, approach the equilibrium solution as $t \to \infty$, then this equilibrium solution is called a <u>stable</u> solution !!!

* The terminal velocity is a stable solution to $v' = -\frac{kv}{m} - g$.

Example 2.9.6

Question is:
Which of these equilibrium solutions is stable???...

Look at next page!



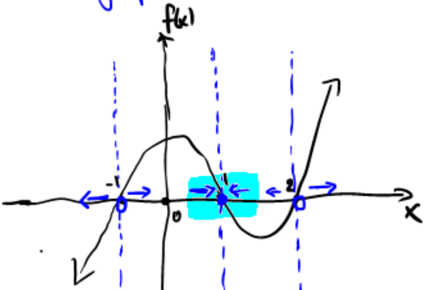
Ex.: $x' = \underbrace{(x^2-1)(x-2)} = f(x)$

$$f(x) = (x-1)(x+1)(x-2) = 0$$

$$\Rightarrow \; x_1 = 1, \; x_2 = -1, \; x_3 = 2$$

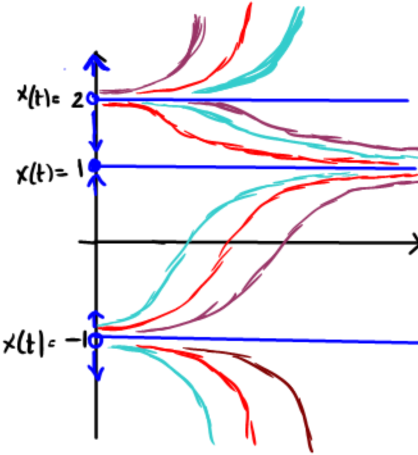
are equilibrium points

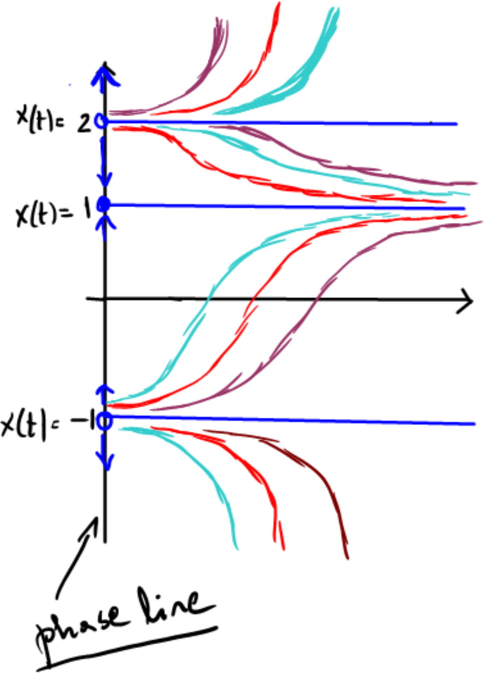
Let's graph the derivative $f(x) = (x-1)(x+1)(x-2)$



$x < -1$	$-1 < x < 1$	$1 < x < 2$	$x > 2$
$f(x) < 0$	$f(x) > 0$	$f(x) < 0$	$f(x) > 0$
$x(t)$ decreasing ←	$x(t)$ increasing →	$x(t)$ decreasing ←	$x(t)$ increasing →

Stable solution



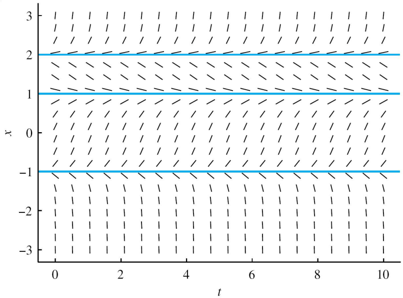


Equilibrium solutions divide the plane into "horizontal" funnels.

In each funnel, the solutions $x(t)$ are either increasing ($f(x) > 0$), or decreasing ($f(x) < 0$).

The rest of slides goes over these above mentioned facts through different examples!

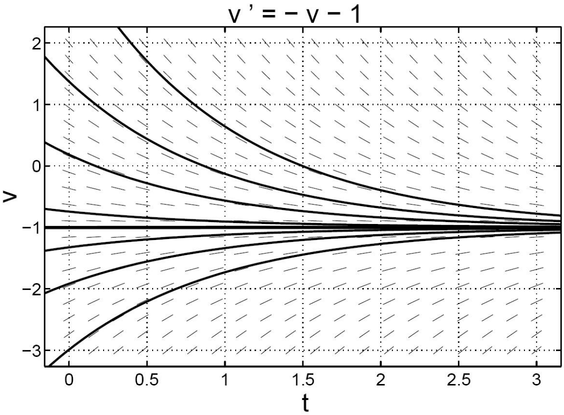
Direction Field and Stability of Equilibrium



- **Direction Field:** same slopes on horizontal lines
- **Equilibrium Solutions $\mathbf{x(t) = x_0}$:**

  $f(x_0) = 0 \Rightarrow$ solution curves are horizontal line
- **Stability of Equilibrium:** Equilibrium point x_0 is
  - *asymptotically stable* if $x(t) \to x_0$ for $t \to \infty$ when $|x(0) - x_0|$ is sufficiently small
  - *unstable* if there are solutions $x(t)$ with $|x(0) - x_0|$ arbitrarily small that move away from x_0 when t increases

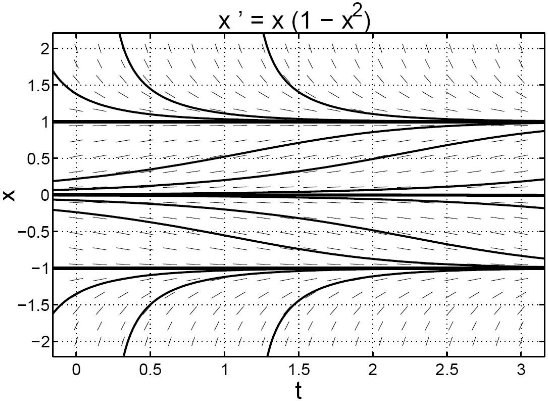
Example: Falling Object and Terminal Velocity



Ex.: $v' = -v - 1$

$v_0 = -1$: asymptotically stable

Example

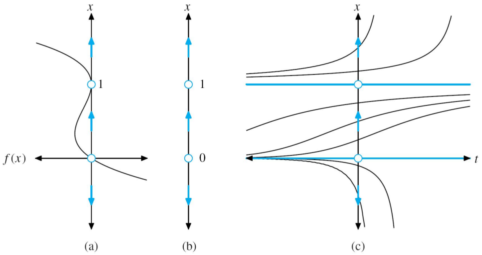


Ex.: $x' = x(1-x^2)$

$x_1 = 0$: unstable

$x_{2,3} = \pm 1$: asymptotically stable

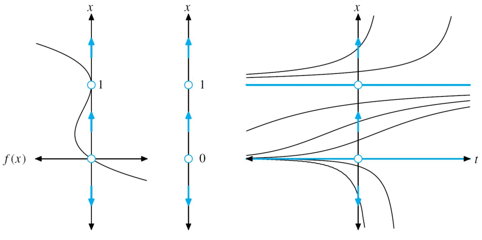
Properties of Solutions



Properties of Solutions

- Equilibrium solutions divide tx–plane into horizontal funnels
- In each funnel solutions are
  -increasing if $x' = f(x) > 0$
  -decreasing if $x' = f(x) < 0$

Phase Line Plots

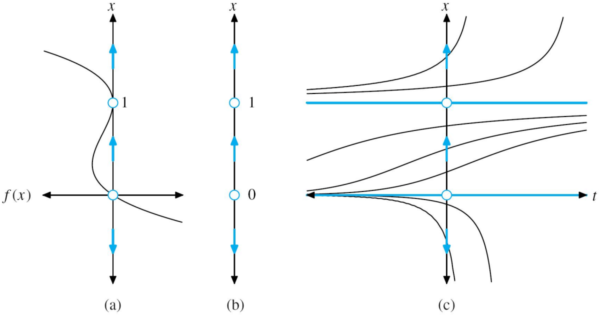


(a) (b) (c)

Phase Line Plots

- Sketch graph $f(x)$ versus x
- Mark equilibrium points on x–axis
- Indicate direction of motion ($x(t)$ decreasing or increasing) by arrows
- Use this to sketch solutions

Stability Criteria



Stability Criteria

Equilibrium point x_0 is

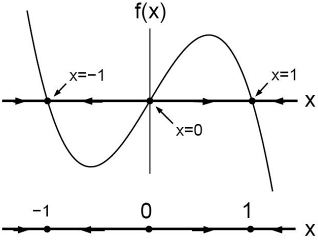
- asympt. stable if $f'(x_0) < 0$
- unstable if $f'(x_0) > 0$

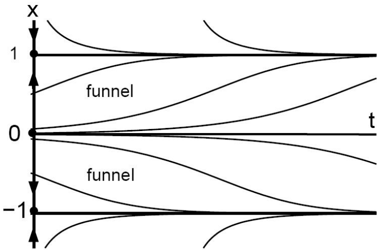
If $f'(x_0) = 0$ inspect graph

Example

Ex.: $x' = x - x^3 = x(1-x)(1+x)$

- $f(x) = 0 \Rightarrow x = 0,\ 1,\ -1$
- $f'(0) = 1 \Rightarrow 0$ is unstable
- $f'(\pm 1) = -1 \Rightarrow \pm 1$ are as. stable



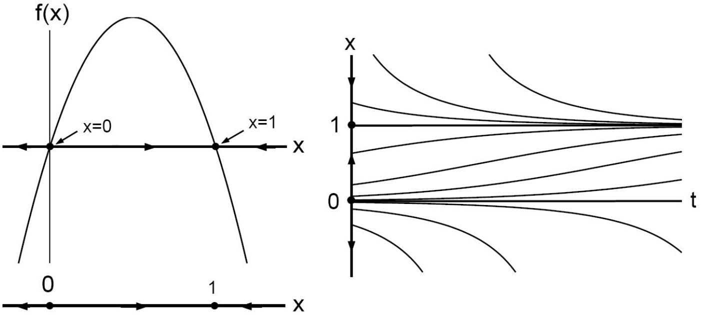


Example

Ex.: $x' = x - x^2 = x(1-x)$

Equilibria:

- $x = 0 \Rightarrow f'(0) = \ \ 1 \Rightarrow$ unstable
- $x = 1 \Rightarrow f'(1) = -1 \Rightarrow$ as. stable



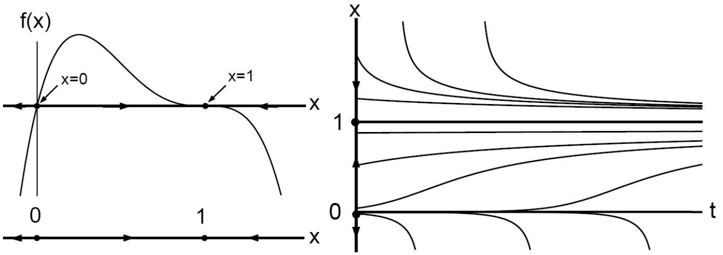
Example

Ex.: $x' = x(1-x)^3$

Equilibria:

- $x = 0 \Rightarrow f'(0) = 1 \Rightarrow$ unstable
- $x = 1 \Rightarrow f'(1) = 0 \Rightarrow$??

  Graph $\Rightarrow$ asympt. stable



Example

Ex.: $x' = -x(1-x)^2$

Equilibria:

- $x = 0 \Rightarrow f'(0) = -1 \Rightarrow$ as. stable
- $x = 1 \Rightarrow f'(1) = \ \ 0 \Rightarrow$??
- Inspect graph: $\Rightarrow x = 1$ is as. stable on right side, unstable on left side (semistable)