Math 3331 Differential Equations

4.1 Second-Order Equations

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4.1 Second-Order Equations

- Second-Order Equation: Models
  - Vibrating Spring
  - Vibrating Spring with Damping
- General Solution
  - Solution Structure
  - Linear Independence and Wronskian
  - Existence and Uniqueness
- Worked out Examples from Exercises
  - 2, 4, 22, 24

Definition

$y'' + t \cdot y' - t^2 y = 5$

Second-Order Equation

$$y'' = f(t, y, y')$$

Linear Equation

$$y'' + p(t)y' + q(t)y = g(t)$$

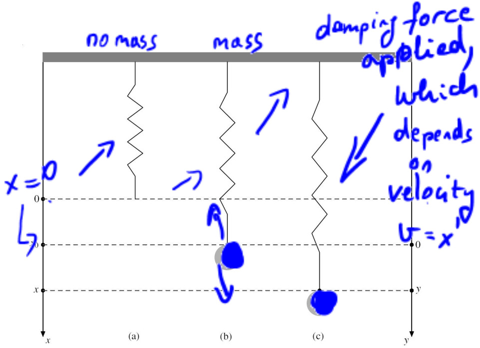
where the coefficients $p(t)$, $q(t)$ and $g(t)$ are functions of t.

Homogeneous Equation

$$y'' + p(t)y' + q(t)y = 0$$

that is, the forcing term $g(t)$ is equal to 0.

Example: Vibrating Spring



Newton's second law:

$$mx'' = mg + R(x) + D(x') + F(t)$$

where

- mg is the force of gravity,
- $R(x)$ the restoring force of the spring,
- $D(x')$ a damping force, and
- $F(t)$ is an external force.

Let $y = x - x_0$ the displacement.

$$\boxed{my'' = -ky + D(y') + F(t)}$$

- Hooke's low: $R(x) = -kx$ with k the spring constant.
- Spring-mass equilibrium: $R(x_0) + mg = 0$.

$m \cdot g = -k \cdot x_0$

Example: Vibrating Spring with Damping

Let the damping force

$$D(y') = -\mu y'$$

with μ the dampling constant.

The 2nd order linear DE for y

$$my'' + \mu y' + ky = F(t)$$

For undamped $\mu = 0$ and unforced $F(t) = 0$ spring, the DE reduces to the harmonic equation

Special case

$$\boxed{y'' + \omega_0^2 y = 0}$$

with $\omega_0 = \sqrt{k/m}$ the natural frequency.

The general solution to the harmonic equation is

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

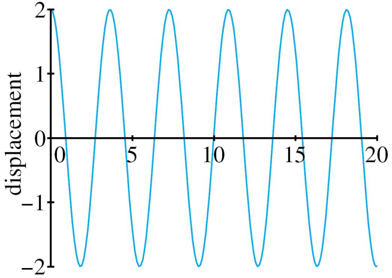


Figure 2 A vibrating spring with no damping.

"No damping force & no external force applied" give rise to a "harmonic motion" situation.

$$my'' + ky = 0$$

$$y'' + \frac{k}{m}y = 0 \quad , \text{ denote } \underbrace{\omega_0 = \sqrt{\frac{k}{m}}}_{\text{natural frequency}}$$

$$\rightarrow \boxed{y'' + \omega_0^2 y = 0}$$

General solution is

$$y = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t).$$

Fast-Overview: $y'' = -\omega_0^2 y$

$y_1(t) = \cos(\omega_0 t)$ possible solution

$y_2(t) = \sin(\omega_0 t)$ possible solution

*

* both of them independent of each other.

general solution $y(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$

$\rightarrow y'(t) = -\omega_0 C_1 \sin(\omega_0 t) + \omega_0 C_2 \cos(\omega_0 t)$

$\Rightarrow y''(t) = -\omega_0^2 C_1 \cos(\omega_0 t) + \omega_0^2 C_2 \sin(\omega_0 t)$

$= -\omega_0^2 \left(C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \right)$

$= -\omega_0^2\, y(t)$. $\square$

Structure of the General Solution

Theorem 1.23

Suppose that y_1 and y_2 are linearly independent solutions to the equation

$$y'' + p(t)y' + q(t)y = 0.$$

Its general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

where C_1 and C_2 are arbitrary constants.

It can be shown that

$$y_1(t) = \cos(\omega_0 t) \text{ and } y_2(t) = \sin(\omega_0 t)$$

are linearly independent solutions to the harmonic equation

$$y'' + \omega_0^2 y = 0$$

Theorem 1.23.

$$y'' + p(t)y' + q(t)y = 0$$

Since $y_1(t)$ and $y_2(t)$ are solutions, then

$$* \begin{cases} y_1'' + p(t)y_1' + q(t)y_1 = 0 \\ y_2'' + p(t)y_2' + q(t)y_2 = 0 \end{cases}$$

Let $y(t) = c_1 y_1 + c_2 y_2$, then check

$$(c_1 y_1'' + c_2 y_2'') + p(t)(c_1 y_1' + c_2 y_2') + q(t)(c_1 y_1 + c_2 y_2) \stackrel{?}{=} 0$$

$\Rightarrow$ regroup

$$c_1(y_1'' + p(t)y_1' + q(t)y) + c_2(y_2'' + p(t)y_2' + q(t)y_2) = 0 \checkmark$$

Thus, to find the solution of

$$y'' + p(t)y' + q(t)y = 0,$$

it's enough to find these two special linearly independent solutions $y_1(t)$ and $y_2(t)$.

How to decide whether y_1 and y_2 are linearly independent ???

Linear independence:

- $\begin{cases} y_1(t) = \sin(t) \\ y_2(t) = \cos(t) \end{cases}$ ✓ can't be multiple of each other

- $\begin{cases} y_1(t) = t \\ y_2(t) = t^2 \end{cases}$ ✓

- $\begin{cases} y_1(t) = t \\ y_2(t) = -4t \end{cases}$ $y_2 = -4y_1$ ✗

y₁ and y₂ depend on each other.

Look at a matrix:

$$A = \left[\begin{array}{c|c} 2 & 6 \\ 4 & 7 \end{array}\right]$$

$$= \left[\begin{array}{c|c} c_1 & c_2 \end{array}\right]$$

- in matrix A, both columns are linearly independent of each other,

$$\det(A) \neq 0$$

linearly independent

$$B = \left[\begin{array}{c|c} 2 & 6 \\ 4 & 12 \end{array}\right]$$

$$B = \left[\begin{array}{c|c} c_1 & 3c_1 \end{array}\right]$$

- in matrix B, the 2nd column is 3 times the first, therefore

$$\det(B) = 0$$

linearly dependent

Linear Independence and Wronskian

Definition 1.22

Two functions u and v are linearly independent on the interval (α, β) if neither is a constant multiple of the other on that interval.

Proposition 1.27

Suppose that u and v are solutions to the equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β). Then u and v are linearly independent if and only if their Wronskian

How do we get it ?

$$W(t) = \det \begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix} = u(t)v'(t) - v(t)u'(t)$$

never vanishes in (α, β), i.e., $W(t_0) \neq 0$ for some t_0 in (α, β).

Having $y'' + p(t)y' + q(t)y = 0$, where
u and v are two linearly independent solutions,

$\Rightarrow$ $\boxed{y(t) = C_1 u + C_2 v}$ must be a general solution

IVP $\quad y'' + p(t)\,y' + q(t)\,y = 0,$

$y(t_0) = y_0 \quad , \quad y'(t_0) = y_1$

$\Rightarrow$ This solution exists if and only if

$$\begin{cases} y(t_0) = C_1 u_0 + C_2 v_0 = y_0 \\ y'(t_0) = C_1 u_0' + C_2 v_0' = y_1 \end{cases}$$

$$\begin{cases} C_1 u(t_0) + C_2 v(t_0) = y_0 \\ C_1 u'(t_0) + C_2 v'(t_0) = y_1 \end{cases}$$ has a solution

$$\Longleftrightarrow$$

$$\begin{bmatrix} u(t_0) & v(t_0) \\ u'(t_0) & v'(t_0) \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$\Longleftrightarrow$$

$$\det \begin{pmatrix} u(t_0) & v(t_0) \\ u'(t_0) & v'(t_0) \end{pmatrix} \neq 0 .$$

the Wronskian of solutions u and v.

$\Longleftrightarrow$ solutions $u(t)$ and $v(t)$ are linearly independent.

IVP and EUT

Theorem 1.17 (Existence and Uniqueness of Solution)

Suppose that $p(t)$,$q(t)$, and $g(t)$ are continuous on (α, β). Let $t_0 \in (\alpha, \beta)$. Then for any real numbers y_0 and y_1, there is one and only one function $y(t)$ defined on (α, β), which is a solution to the the initial value problem

$$y'' + p(t)y' + q(t)y = g(t) \quad \text{for } \alpha < t < \beta$$

with the initial conditions

$$y(t_0) = y_0, \quad \text{and} \quad y'(t_0) = y_1.$$

Example 1.31

Example

Find the solution to the harmonic equation $x'' + 4x = 0$ with intial conditions $x(0) = 4$ and $x'(0) = 2$.

$x_1(t) = \cos(2t)$, $x_2(t) = \sin(2t)$ ← linearly ind.

We know from Example 1.24 that the general solution has the form

general solution →
$$x(t) = a\cos 2t + b\sin 2t,$$

where a and b are arbitrary constants. Substituting the initial conditions we get

initial condition →
$$4 = x(0) = a, \quad \text{and} \quad 2 = x'(0) = 2b.$$

Thus $a = 4$ and $b = 1$ and our solution is

$$\boxed{x(t) = 4\cos 2t + \sin 2t.}$$

Exercise 4.1.2

Determine whether the equation

$$t^2y'' = 4y' - \sin t$$

is linear or nonlinear. If linear, state whether it is homogeneous or inhomogeneous.

Divide both sides of $t^2y'' = 4y' - \sin t$ by t^2, then rearrange to obtain

$$y'' - \frac{4}{t^2}y' = -\frac{\sin t}{t^2}.$$

Compare this with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = -4/t^2$, $q(t) = 0$, and $g(t) = -(\sin t)/t^2$. Hence, the equation is linear and inhomogeneous.

Exercise 4.1.4

Determine whether the equation

$$ty'' + (\sin t)\, y' = 4y - \cos 5t$$

is linear or nonlinear. If linear, state whether it is homogeneous or inhomogeneous.

Divide both sides of $ty'' + (\sin t)y' = 4y - \cos 5t$ by t, then rearrange to obtain

$$y'' + \frac{\sin t}{t} y' - \frac{4}{t} = -\frac{\cos 5t}{t}$$

Compare this with

$$y'' + p(t)y' + q(t)y = g(t),$$

and note that $p(t) = (\sin t)/t$, $q(t) = -4/t$, and $g(t) = -(\cos 5t)/t$. Hence, the equation is linear and inhomogeneous.

Exercise 4.1.22

Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for

$$y'' + 2y' - 3y = 0,$$

then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.

If $y_1(t) = e^t$, then

$$y'' + 2y' - 3y = e^t + 2e^t - 3e^t = 0,$$

and if $y_2(t) = e^{-3t}$, then

$$y'' + 2y' - 3y = 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0,$$

Furthermore,

$$\frac{y_1(t)}{y_2(t)} = \frac{e^t}{e^{-3t}} = e^{4t},$$

which is nonconstant. Thus, y_1 is not a constant multiple of y_2 and the solutions $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions.

Thus, the general solution of $y'' + 2y' - 3y = 0$ is

$$y(t) = C_1e^t + C_2e^{-3t},$$

and its derivative is

$$y'(t) = C_1e^t - 3C_2e^{-3t}t.$$

The initial conditions, $y(0) = 1$ and $y'(0) = -2$ lead to the equations

$$1 = C_1 + C_2$$
$$-2 = C_1 - 3C_2$$

and the constants $C_1 = 1/4$ and $C_2 = 3/4$. Thus, the solution of the initial value problem is

$$y(t) = \frac{1}{4}e^t + \frac{3}{4}e^{-3t}.$$

Exercise 4.1.24

Show that $y_1(t) = e^{-t}\cos 2t$ and $y_2(t) = e^{-t}\sin 2t$ form a fundamental set of solutions for

$$y'' + 2y' + 5y = 0,$$

then find a solution satisfying $y(0) = -1$ and $y'(0) = 0$.

If $y_1(t) = e^{-t}\cos 2t$, then

$$y_1'(t) = -e^{-t}\cos 2t - 2e^{-t}\sin 2t, \quad \text{and}$$
$$y_1''(t) = -3e^{-t}\cos 2t + 4e^{-t}\sin 2t.$$

Thus,

$$\begin{aligned} & y_1'' + 2y_1' + 5y_1 \\ &= -3e^{-t}\cos 2t + 4e^{-t}\sin 2t \\ &\quad - 2e^{-t}\cos 2t - 4e^{-t}\sin 2t + 5e^{-t}\cos 2t \\ &= 0. \end{aligned}$$

If $y_2(t) = e^{-t}\sin 2t$, then

$$y_2'(t) = -e^{-t}\sin 2t + 2e^{-t}\cos 2t, \quad \text{and}$$
$$y_2''(t) = -3e^{-t}\sin 2t - 4e^{-t}\cos 2t.$$

Thus,

$$\begin{aligned} & y_2'' + 2y_2' + 5y_2 \\ &= -3e^{-t}\sin 2t - 4e^{-t}\cos 2t \\ &\quad - 2e^{-t}\sin 2t + 4e^{-t}\cos 2t + 5e^{-t}\sin 2t \\ &= 0. \end{aligned}$$

Furthermore,

$$\frac{y_1(t)}{y_2(t)} = \frac{e^{-t}\cos 2t}{e^{-t}\sin 2t} = \cot 2t,$$

which is nonconstant. Thus, y_1 is not a constant multiple of y_2 and the solutions $y_1(t) = e^{-t}\cos 2t$ and $y_2(t) = e^{-t}\sin 2t$ form a fundamental set of solutions. Thus, the general solution of $y''+2y'+5y = 0$ is

$$y(t) = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t,$$

and its derivative is

$$\begin{aligned} y'(t) &= -C_1 e^{-t}\cos 2t - 2C_1 e^{-t}\sin 2t \\ &\quad - C_2 e^{-t}\sin 2t + 2C_2 e^{-t}\cos 2t. \end{aligned}$$

The initial conditions, $y(0) = -1$ and $y'(0) = 0$ lead to the equations

$$\begin{aligned} -1 &= C_1 \\ 0 &= -C_1 + 2C_2 \end{aligned}$$

and the constants $C_1 = -1$ and $C_2 = -1/2$. Thus, the solution of the initial value problem is

$$y(t) = -e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t.$$