Math 3331 Differential Equations

4.2 Second-Order Equations and Systems

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4.2 Second-Order Equations and Systems

- Second-Order Equations
- Planar Systems
  - *yv*-Phase Plane Plot
  - Phase Plane Portrait

<u>Definition</u>: A <u>planar system</u> of 1^{st} order diff. equations is a <u>set of two</u> 1^{st} order diff. equations involving two unknown functions.

<u>eg</u>
$$\begin{cases} x' = f(t, x, y) \\ y' = g(t, x, y) \end{cases}$$

f, g are functions of independent variable t and two unknowns x and y.

- Explore connections btw higher order DE and 1st order systems.
- Explore ways to visualize solutions to 2^{nd} order DE.

1st order DE

$y' = f(v)$

- Use of phase plane

$\downarrow$

solutions to planar systems of DE.

We'll begin with 2^{nd} order DE:

$$(*) \qquad y'' = F(t, y, y')$$

$\Rightarrow$ introduce $\begin{cases} v = y' \\ v' = F(t, y, v) \end{cases}$

If $y(t)$ is a solution of $(*)$:

$$y'' = v' = F(t, y, v) = F(t, y, y') \qquad \square$$

ex

$$y'' + p(t)y' + q(t)y = g(t) \quad (*)$$

$$\Rightarrow \quad v = y'$$

From $(*)$, $\quad y'' = -p(t)y' - q(t)y + g(t)$

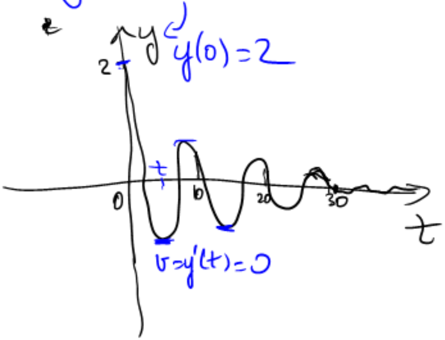
$$= -p(t)v - q(t)y + g(t)$$

$$= F(t, y, v)$$

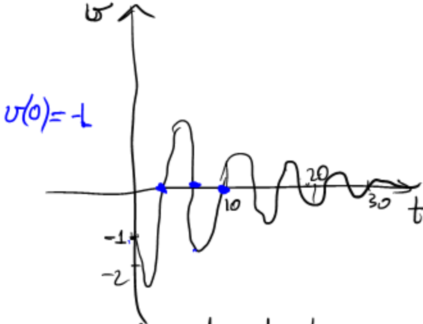
$$\Rightarrow \begin{cases} y' = v \\ v' = g(t) - p(t)v - q(t)y. \end{cases} \quad (*)$$

Visualization of solutions. ~ Graph

$y' = v \leftarrow$ displacement



graph of displacement



graph of velocity.

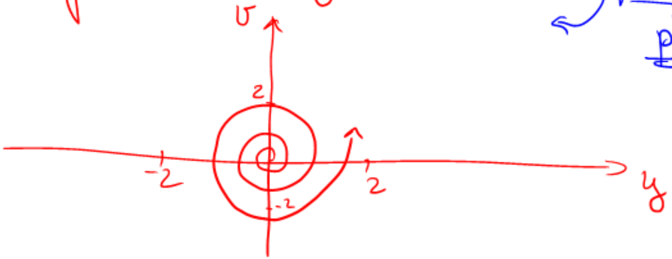
ex $\quad my'' + \mu y' + ky = 0$

$\hookrightarrow \quad y'' + 0.4\,y' + 3y = 0 \quad , \; y(0) = 2$

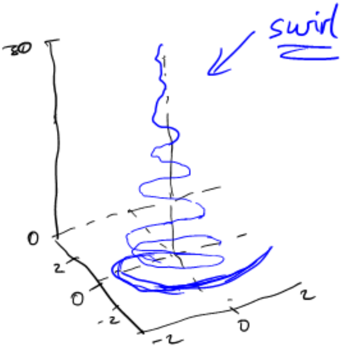
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y'(0) = -1 = v(0)$

The pictures resembles the one in previous slide.

Put together in (yv)-plane: phase plane plot



in 3D:



Second-Order Equations and Planar Systems

Second-order DE

$$y'' + ay' + by = 0 \qquad (1)$$

$$p(\lambda) = \lambda^2 + a\lambda + b = 0$$

planar system

$$x_1 = y, \quad x_2 = v = y'$$

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \qquad (2)$$

$$\det(A - \lambda I) = p(\lambda)$$

(Chapter 9)

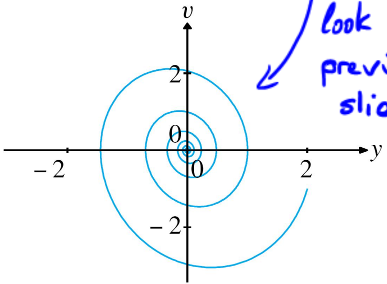
yv-Phase Plane Plot

A damped unforced spring:

$$my'' + \mu y' + ky = 0$$

with $m = 1$, $\mu = 0.4$, and $k = 3$.

look at previous slides



(Spi

sink)

Phase Plane Portrait

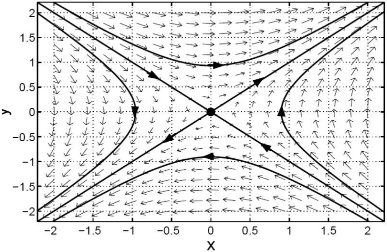
Ex.: $y'' - y = 0 \;(a = 0, b = -1)$

$$p(\lambda) = \lambda^2 - 1 \Rightarrow \lambda = \pm 1 \text{ (saddle)}$$

details will be covered in Chapter 9.

General solution: $y(t) = c_1 e^t + c_2 e^{-t}$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to \begin{cases} \lambda_1 = 1 & \leftrightarrow \mathbf{v}_1 = [1, 1]^T \\ \lambda_2 = -1 & \leftrightarrow \mathbf{v}_2 = [-1, 1]^T \end{cases}$$



Phase plane portrait for DE (1) = Phase plane portrait for (2)

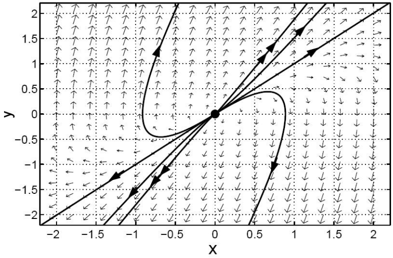
Phase Plane Portrait

Ex.: $y'' - 3y' + 2y = 0$

$$p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

$\Rightarrow$ source: $y(t) = c_1 e^t + c_2 e^{2t}$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \rightarrow \begin{cases} \lambda_1 = 1 \leftrightarrow \mathbf{v}_1 = [1, 1]^T \\ \lambda_2 = 2 \leftrightarrow \mathbf{v}_2 = [1, 2]^T \end{cases}$$



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(Chapter 9)