Math 3331 Differential Equations 4.3 Linear, Homogeneous Equations with Constant Coefficients

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4.3 Linear, Homogeneous Equations with Constant Coefficients

- Definition and Key Idea
- DE and its Characteristic Equation
- Characteristic Roots and General Solution
 - Distinct Real Roots
 - Complex Roots
 - Repeated Roots
- Worked out Examples from Exercises
 - Distinct Real Roots: 2, 25
 - Complex Roots: 10
 - Repeated Roots: 18



The Key Idea

Linear, Homogeneous Equations with Constant Coefficients

$$y'' + \underline{p}y' + \underline{q}y = 0$$

where p and q are constant.

The Key Idea

Look for a solution of the type $y(t) = e^{\lambda t}$ where λ is a constant, as yet unknown. Inserting it into the DE,

$$y'' + py' + qy = \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} = (\lambda^2 + p\lambda + q)e^{\lambda t} = 0.$$

Since $e^{\lambda t} \neq 0$, then

$$\lambda^2 + p\lambda + q = 0$$

This is called the characteristic equation for the DE.

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Characteristic Root

DE and its Characteristic Equation

$$y'' + py' + qy = 0$$
$$\lambda^2 + p\lambda + q = 0$$

Characteristic Root

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

- two distinct real roots if $p^2 4q > 0$.
- two distinct complex roots if $p^2 4q < 0$.
- one repeated real root if $p^2 4q = 0$.

Distinct Real Roots

Proposition 3.3

If the characteristic equation $\lambda^2 + p\lambda + q = 0$ has two distinct real roots λ_1 and λ_2 , then the general solution to y'' + py' + qy = 0 is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

where C_1 and C_2 are arbitrary constants.

IVP

The particular solution for an initial value problem can be found by evaluating the constants C_1 and C_2 using the initial conditions.



Example 3.4

Find the general solution to the equation

$$y'' - 3y' + 2y = 0.$$

Find the unique solution corresponding to the initial conditions y(0) = 2and y'(0) = 1. (Ans: $y(t) = -e^{2t} + 3e^t$)

DE, its Characteristic Equation and roots

$$y'' - 3y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 - 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = 2, \lambda_2 = 1$$

The general solution is

 $y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{2t} + C_2 e^t \quad \Rightarrow y'(t) = 2C_1 e^{2t} + C_2 e^t$

ICs: $y(0) = 2 = C_1 + C_2$ and $y'(0) = 1 = 2C_1 + C_2$ imply

$$C_1 = -1, C_2 = 3 \quad \Rightarrow y(t) = -e^{2t} + 3e^t$$

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ylt) = e^{xt} y'' - 3y' + 2y = 0 $\int_{2}^{2} \lambda^{2} - 3\lambda + 2 = 0$ =(2-1)(2-2)=0 =) 1=1, 2 2=2 $y(t) = e^t$, $y_2(t) = e^{t}$ =)/y(t) = $C_1 e^{t} + C_2 e^{t} = y(0) = 2$, y'(0) = 1 $(y't) = C_1 e^{t} + 2C_2 e^{2t}$ -=> &= -1 $= \begin{array}{c} y_{(0)} = (c_1 + c_2 = 2) \\ y_{(0)} = (c_1 + 2(c_2 = 1)) \end{array}$ C_= 3 \Rightarrow y(l) = $3e_{1}^{2} - e_{1}^{2t}$

Complex Roots

Proposition 3.20

Suppose the characteristic equation $\lambda^2 + p\lambda + q = 0$ has two complex conjugate roots $\lambda = a + ib$ and $\overline{\lambda} = a - ib$.

1. The functions

$$z(t) = e^{\lambda t} = e^{(a+ib)t}$$
 and $\overline{z}(t) = e^{\overline{\lambda}t} = e^{(a-ib)t}$

form a complex valued fundamental set of solutions, so the general solution the general solution to y'' + py' + qy = 0 is

$$y(t) = C_1 e^{\lambda t} + C_2 e^{\bar{\lambda}t} = C_1 e^{(a+ib)t} + C_2 e^{(a-ib)t}$$

where C_1 and C_2 are arbitrary complex constants.

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$$\sum_{x = 0}^{\infty} \lambda^{2} + \rho\lambda + q = 0$$

$$\lambda = a + ib \qquad \overline{\lambda} = a - ib$$

$$= e^{at} = e^{(a + ib)T} = e^{at} (\cos(bt) + i\sin(bt))$$

$$= e^{at} \cdot e^{ibt} = e^{at} (\cos(bt) + i\sin(bt))$$

$$= e^{at} \cdot e^{ibt}$$

$$y_{1}(t) = e^{at} \cos(bt) + i e^{at} \sin(bt)$$

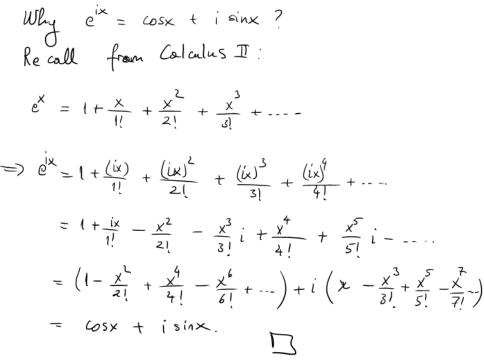
$$y_{2}(t) = e^{at} \cos(bt) + i e^{at} \sin(bt)$$

$$y_{1}(t) = e^{at} \cos(bt) + i e^{at} \sin(bt)$$

$$y_{2}(t) = e^{at} \cos(bt) + i e^{at} \sin(bt)$$

$$y_{2}(t) = e^{at} \cos(bt) + i e^{at} \sin(bt)$$

(et cos(b)) i e sin(bt) $\frac{\partial f^2}{\partial z} =$ ylt) = C1 ent cos(bt) + (2. i ent sin(bt) constant $\lambda = 2 + \frac{1}{2}$ Fundamented set: $y_1 = e^{zt} \cos(zt)$ $y_2 = e^{zt} \sin(zt)$ => $y(t) = c_1 e^{it} cos(st) + C_2 e^{2t} sin(st)$.



Complex Roots (cont.)

Proposition 3.20 (cont.)

2. The functions

$$y_1(t) = e^{at} \cos(bt)$$
 and $y_2(t) = e^{at} \sin(bt)$

form a real valued fundamental set of solutions, so the general solution the general solution to y'' + py' + qy = 0 is

$$y(t) = e^{at}(A_1\cos(bt) + A_2\sin(bt)),$$

where A_1 and A_2 are arbitrary real constants.

Real and Imaginary Parts

$$z(t) = y_1(t) + iy_2(t), \quad \bar{z}(t) = y_1(t) - iy_2(t)$$

$$y_1(t) = \operatorname{Re} z(t) = \frac{1}{2}(z(t) + \bar{z}(t)), \quad y_2(t) = \operatorname{Im} z(t) = \frac{1}{2i}(z(t) - \bar{z}(t))$$

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Example 3.21

Find the general solution to the equation

$$y'' + 2y' + 2y = 0.$$

Find the unique solution corresponding to the initial conditions y(0) = 2and y'(0) = 3. (Ans: $y(t) = e^{-t}(2\cos t + 5\sin t)$)

DE, its Characteristic Equation and roots

$$y'' + 2y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -1 \pm i$$

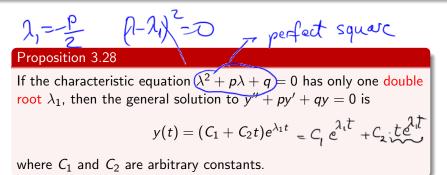
$$y(t) = e^{at}(C_1 \cos(bt) + C_2 \sin(bt)) = e^{-t}(C_1 \cos(t) + C_2 \sin(t))$$

$$\Rightarrow y'(t) = -e^{-t}(C_1 \cos(t) + C_2 \sin(t)) + e^{-t}(-C_1 \sin(t) + C_2 \cos(t))$$

ICs: $y(0) = 2 = C_1$ and $y'(0) = 3 = -C_1 + C_2$ imply
 $C_1 = 2, C_2 = 5 \Rightarrow y(t) = e^{-t}(2\cos(t) + 5\sin(t))$



Repeated Roots



IVP

The particular solution for an initial value problem can be found by evaluating the constants C_1 and C_2 using the initial conditions.



$$y'' + py' + qy = 0$$

$$y'' + py' + qy = 0$$

$$y_1(t) = e^{2t} = e^{2t}$$

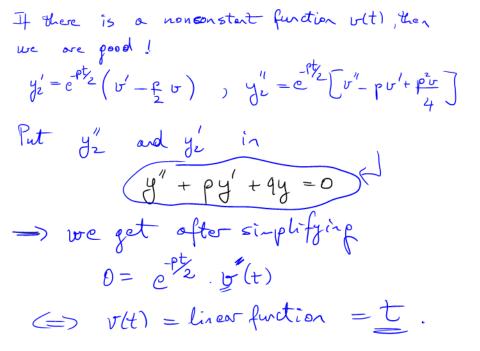
$$y_1(t) = e^{2t} = e^{2t}$$

$$double root$$

$$A_1 = \frac{-p}{2}$$

$$A_1 = \frac{-p}{2}$$

$$A_2 = \frac{-p}{2}$$



 \Rightarrow $y_2(t) = te^{-pt_2}$

General solution: $y(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$ R, = repeated root!

Example 3.29

Find the general solution to the equation

$$y''-2y'+y=0.$$

Find the unique solution corresponding to the initial conditions y(0) = 2and y'(0) = -1. (Ans: $y(t) = 2e^t - 3t e^t$)

DE, its Characteristic Equation and roots

$$y'' - 2y' + y = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 1$$

$$y(t) = (C_1 + C_2 t)e^{\lambda_1 t} = (C_1 + C_2 t)e^t$$
$$\Rightarrow y'(t) = (C_1 + C_2 t)e^t + C_2 e^t$$
$$Cs: y(0) = 2 = C_1 \text{ and } y'(0) = -1 = C_1 + C_2 \text{ imply}$$

$$C_1=2, C_2=-3 \Rightarrow y(t)=(2-3t)e^t.$$



 $\int_{3}^{3} y(0) = 2 + y = 0$ => 2²-22+1-0 2=1 double value => $y_1(t) = e^{t}$ => $y_2(t) = t \cdot e^{t}$ $y_1(t) = C_1 \cdot e^{t} + C_2 t \cdot e^{t}$ $y_2(t) = t \cdot e^{t}$ $y_1(t) = C_1 \cdot e^{t} + C_2 (e^{t} + t \cdot e^{t})$ $= 2 = C_{1} + C_{2} \quad (-1) = C_{1} + C_{2} \quad (-2) = 2 \quad (-2) = -3$ $= \gamma y(t) = 2e^{t} - 3te^{t}$

Exercise 4.3.2

Find the general solution to the equation y'' + 5y' + 6y = 0.

DE, its Characteristic Equation and roots

$$y'' + 5y' + 6y = 0 \quad \Rightarrow \quad \lambda^2 + 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda_1 = -3, \lambda_2 = -2$$

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-3t} + C_2 e^{-2t}$$



Exercise 4.3.10

Find the general solution to the equation y'' + 4 = 0.

DE, its Characteristic Equation and roots

$$y'' + 4 = 0 \quad \Rightarrow \quad \lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 2i$$

$$y(t) = e^{at}(C_1\cos(bt) + C_2\sin(bt)) = C_1\cos(2t) + C_2\sin(2t)$$



Exercise 4.3.18

Find the general solution to the equation y'' - 6y' + 9y = 0.

DE, its Characteristic Equation and roots

$$y'' - 6y' + 9y = 0 \quad \Rightarrow \quad \lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 3$$

The general solution is

$$y(t) = (C_1 + C_2 t)e^{\lambda_1 t} = (C_1 + C_2 t)e^{3t}$$

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Exercise 4.3.25

Find the solution of the initial value problem y'' - y' - 2y = 0, y(0) = -1, y'(0) = 2.

DE, its Characteristic Equation and roots

$$y'' - y' - 2y = 0 \quad \Rightarrow \quad \lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \quad \lambda_1 = 2, \lambda_2 = -1$$

The general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{2t} + C_2 e^{-t} \quad \Rightarrow y'(t) = 2C_1 e^{2t} - C_2 e^{-t}$$

ICs: $y(0) = -1 = C_1 + C_2$ and $y'(0) = 2 = 2C_1 - C_2$ imply
 $C_1 = 1/3, C_2 = -4/3 \quad \Rightarrow y(t) = \frac{1}{3}e^{2t} - \frac{4}{3}e^t$

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