Math 3331 Differential Equations

4.3 Linear, Homogeneous Equations with Constant Coefficients

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4.3 Linear, Homogeneous Equations with Constant Coefficients

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The Key Idea

Linear, Homogeneous Equations with Constant Coefficients

$$y'' + py' + qy = 0$$

where p and q are constant.

$y' = k \cdot y$

$y(t) = Ce^{kt}$

The Key Idea

Look for a solution of the type $y(t) = e^{\lambda t}$ where λ is a constant, as yet unknown. Inserting it into the DE,

$$y'' + py' + qy = \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} = (\lambda^2 + p\lambda + q)e^{\lambda t} = 0.$$

Since $e^{\lambda t} \neq 0$, then

$$\lambda^2 + p\lambda + q = 0$$

This is called the characteristic equation for the DE.

Characteristic Root

DE and its Characteristic Equation

$$y'' + py' + qy = 0$$
$$\lambda^2 + p\lambda + q = 0$$

Characteristic Root

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

- two distinct real roots if $p^2 - 4q > 0$.
- two distinct complex roots if $p^2 - 4q < 0$.
- one repeated real root if $p^2 - 4q = 0$.

Distinct Real Roots

Proposition 3.3

If the characteristic equation $\lambda^2 + p\lambda + q = 0$ has two distinct real roots λ_1 and λ_2, then the general solution to $y'' + py' + qy = 0$ is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

where C_1 and C_2 are arbitrary constants.

IVP

The particular solution for an initial value problem can be found by evaluating the constants C_1 and C_2 using the initial conditions.

Example 3.4

Find the general solution to the equation

$$y'' - 3y' + 2y = 0.$$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = 1$. (Ans: $y(t) = -e^{2t} + 3e^{t}$)

DE, its Characteristic Equation and roots

$$y'' - 3y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 - 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = 2, \lambda_2 = 1$$

The general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{2t} + C_2 e^{t} \quad \Rightarrow y'(t) = 2C_1 e^{2t} + C_2 e^{t}$$

ICs: $y(0) = 2 = C_1 + C_2$ and $y'(0) = 1 = 2C_1 + C_2$ imply

$$C_1 = -1, C_2 = 3 \quad \Rightarrow y(t) = -e^{2t} + 3e^{t}$$

$y'' - 3y' + 2y = 0$ $\qquad y(t) = e^{\lambda t}$

$\hookrightarrow \lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 2$

$y_1(t) = e^t, \quad y_2(t) = e^{2t}$

$\Rightarrow \begin{cases} y(t) = c_1 e^t + c_2 e^{2t} \\ y'(t) = c_1 e^t + 2c_2 e^{2t} \end{cases} \leftarrow \quad y(0) = 2, \quad y'(0) = 1$

$\Rightarrow \begin{array}{l} y(0) = c_1 + c_2 = 2 \\ y'(0) = c_1 + 2c_2 = 1 \end{array} \Rightarrow c_2 = -1, \quad c_1 = 3$

$\Rightarrow y(t) = 3e^t - e^{2t}$ $\square$

Complex Roots

Proposition 3.20

Suppose the characteristic equation $\lambda^2 + p\lambda + q = 0$ has two complex conjugate roots $\lambda = a + ib$ and $\bar{\lambda} = a - ib$.

1. The functions

$$z(t) = e^{\lambda t} = e^{(a+ib)t} \quad \text{and} \quad \bar{z}(t) = e^{\bar{\lambda} t} = e^{(a-ib)t}$$

form a complex valued fundamental set of solutions, so the general solution the general solution to $y'' + py' + qy = 0$ is

$$y(t) = C_1 e^{\lambda t} + C_2 e^{\bar{\lambda} t} = C_1 e^{(a+ib)t} + C_2 e^{(a-ib)t}$$

where C_1 and C_2 are arbitrary complex constants.

$$\Rightarrow \quad \lambda^2 + p\lambda + q = 0$$

$$\lambda = a + ib \qquad \bar{\lambda} = a - ib$$

$$\Rightarrow y_1(t) = e^{\lambda t} = e^{(a+ib)t} = e^{at}\left(\cos(bt) + i\sin(bt)\right)$$

$$= e^{at} \cdot e^{ibt}$$

$$e^{ix} = \cos x + i \sin x \quad (*)$$

$$y_2(t) = e^{(a-ib)t} = e^{at - ibt} = e^{at} \cdot \left(\cos(bt) - i\sin(bt)\right)$$

$$\Rightarrow y_1(t) = e^{at}\cos(bt) + i\, e^{at}\sin(bt)$$

$$y_2(t) = e^{at}\cos(bt) - i\, e^{at}\sin(bt)$$

$$y(t) = C_1 y_1 + C_2 y_2$$

$$\frac{y_1 + y_2}{2} = \left(e^{at}\cos(bt)\right) \qquad \frac{y_1 - y_2}{2} = \left(i\, e^{at}\sin(bt)\right)$$

$$y(t) = C_1 \cdot e^{at}\cos(bt) + \underbrace{C_2 \cdot i}_{\text{constant}}\, e^{at}\sin(bt)$$

ex

$$\lambda = \underset{a}{2} + \underset{b}{3}i$$

Fundamental set: $y_1 = e^{2t}\cos(3t)$

$y_2 = e^{2t}\sin(3t)$

$$\Rightarrow y(t) = C_1 e^{2t}\cos(3t) + C_2 e^{2t}\sin(3t).$$

Why $e^{ix} = \cos x + i \sin x$?

Recall from Calculus II:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\Rightarrow e^{ix} = 1 + \frac{(ix)}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

$$= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{x^3}{3!} i + \frac{x^4}{4!} + \frac{x^5}{5!} i - \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\right)$$

$$= \cos x + i \sin x.$$

$\square$

Complex Roots (cont.)

Proposition 3.20 (cont.)

2. The functions

$$y_1(t) = e^{at}\cos(bt) \quad \text{and} \quad y_2(t) = e^{at}\sin(bt)$$

form a real valued fundamental set of solutions, so the general solution the general solution to $y'' + py' + qy = 0$ is

$$y(t) = e^{at}(A_1\cos(bt) + A_2\sin(bt)),$$

where A_1 and A_2 are arbitrary real constants.

Real and Imaginary Parts

$$z(t) = y_1(t) + iy_2(t), \quad \bar{z}(t) = y_1(t) - iy_2(t)$$

$$y_1(t) = \text{Re } z(t) = \frac{1}{2}(z(t) + \bar{z}(t)), \; y_2(t) = \text{Im } z(t) = \frac{1}{2i}(z(t) - \bar{z}(t))$$

Example 3.21

Find the general solution to the equation

$$y'' + 2y' + 2y = 0.$$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = 3$. (Ans: $y(t) = e^{-t}(2\cos t + 5\sin t)$)

DE, its Characteristic Equation and roots

$$y'' + 2y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -1 \pm i$$

The general solution is

$$y(t) = e^{at}(C_1\cos(bt) + C_2\sin(bt)) = e^{-t}(C_1\cos(t) + C_2\sin(t))$$

$$\Rightarrow y'(t) = -e^{-t}(C_1\cos(t) + C_2\sin(t)) + e^{-t}(-C_1\sin(t) + C_2\cos(t))$$

ICs: $y(0) = 2 = C_1$ and $y'(0) = 3 = -C_1 + C_2$ imply

$$C_1 = 2, C_2 = 5 \quad \Rightarrow y(t) = e^{-t}(2\cos(t) + 5\sin(t))$$

Repeated Roots

$\lambda_1 = -\frac{p}{2}$ $\quad (\lambda - \lambda_1)^2 = 0$ $\quad$ → perfect square

Proposition 3.28

If the characteristic equation $\lambda^2 + p\lambda + q = 0$ has only one double root λ_1, then the general solution to $y'' + py' + qy = 0$ is

$$y(t) = (C_1 + C_2 t)e^{\lambda_1 t} = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$$

where C_1 and C_2 are arbitrary constants.

IVP

The particular solution for an initial value problem can be found by evaluating the constants C_1 and C_2 using the initial conditions.

$$y'' + py' + qy = 0$$

$\hookrightarrow \underbrace{\lambda^2 + p\lambda + q}_{\text{perfect square}} = 0 \implies$ double root $(\lambda - \lambda_1)^2 = 0$

$$\downarrow$$

$$\underbrace{\lambda_1 = \frac{-p}{2}}_{\text{double root}}$$

$$\implies y_1(t) = e^{\lambda_1 t} = e^{-\frac{p}{2}t} \checkmark$$

- We need another independent fundamental solution.

  - Method for building a new solution knowing one solution already:

$$y_2(t) = v(t) \cdot y_1(t)$$

$$= v_t \cdot e^{-\frac{pt}{2}}$$

$$\uparrow$$

If there is a nonconstant function $v(t)$, then we are good!

$$y_2' = e^{-pt/2}\left(v' - \frac{p}{2} v\right), \quad y_2'' = e^{-pt/2}\left[v'' - p v' + \frac{p^2 v}{4}\right]$$

Put y_2'' and y_2' in

$$y'' + p y' + q y = 0$$

$\Longrightarrow$ we get after simplifying

$$0 = e^{-pt/2} \cdot v''(t)$$

$\Longleftrightarrow$ $v(t)$ = linear function $= t$.

$$\Rightarrow \quad y_2(t) = t e^{-pt/2}$$

General solution:

$$y(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$$

λ_1 = repeated root!

Example 3.29

Find the general solution to the equation

$$y'' - 2y' + y = 0.$$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = -1$. (Ans: $y(t) = 2e^t - 3t\, e^t$)

DE, its Characteristic Equation and roots

$$y'' - 2y' + y = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 1$$

The general solution is

$$y(t) = (C_1 + C_2 t)e^{\lambda_1 t} = (C_1 + C_2 t)e^t$$

$$\Rightarrow y'(t) = (C_1 + C_2 t)e^t + C_2 e^t$$

ICs: $y(0) = 2 = C_1$ and $y'(0) = -1 = C_1 + C_2$ imply

$$C_1 = 2, C_2 = -3 \quad \Rightarrow y(t) = (2 - 3t)e^t.$$

$$y'' - 2y' + y = 0 \Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$\hookrightarrow y(0) = 2, \; y'(0) = -1$

$\lambda = 1$ double value

$$\Rightarrow y_1(t) = e^t$$

$$\Rightarrow y_2(t) = t \cdot e^t$$

$$y(t) = C_1 e^t + C_2 t e^t$$

$$y'(t) = C_1 \cdot e^t + C_2 (\underbrace{e^t}_{1} + \underbrace{t \cdot e^t}_{0})$$

$$\Rightarrow \left. \begin{array}{l} 2 = C_1 + C_2 \cdot 0 \\ -1 = C_1 + C_2 \end{array} \right\} \Rightarrow C_1 = 2, \; C_2 = -3$$

$$\Rightarrow y(t) = 2e^t - 3te^t \quad \square$$

Exercise 4.3.2

Find the general solution to the equation

$$y'' + 5y' + 6y = 0.$$

DE, its Characteristic Equation and roots

$$y'' + 5y' + 6y = 0 \quad \Rightarrow \quad \lambda^2 + 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda_1 = -3, \lambda_2 = -2$$

The general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-3t} + C_2 e^{-2t}$$

Exercise 4.3.10

Find the general solution to the equation

$$y'' + 4 = 0.$$

DE, its Characteristic Equation and roots

$$y'' + 4 = 0 \quad \Rightarrow \quad \lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 2i$$

The general solution is

$$y(t) = e^{at}(C_1 \cos(bt) + C_2 \sin(bt)) = C_1 \cos(2t) + C_2 \sin(2t)$$

Exercise 4.3.18

Find the general solution to the equation
$$y'' - 6y' + 9y = 0.$$

DE, its Characteristic Equation and roots

$$y'' - 6y' + 9y = 0 \quad \Rightarrow \quad \lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 3$$

The general solution is

$$y(t) = (C_1 + C_2 t)e^{\lambda_1 t} = (C_1 + C_2 t)e^{3t}$$

Exercise 4.3.25

Find the solution of the initial value problem

$$y'' - y' - 2y = 0, \quad y(0) = -1, \quad y'(0) = 2.$$

DE, its Characteristic Equation and roots

$$y'' - y' - 2y = 0 \quad \Rightarrow \quad \lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \quad \lambda_1 = 2, \lambda_2 = -1$$

The general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{2t} + C_2 e^{-t} \quad \Rightarrow y'(t) = 2C_1 e^{2t} - C_2 e^{-t}$$

ICs: $y(0) = -1 = C_1 + C_2$ and $y'(0) = 2 = 2C_1 - C_2$ imply

$$C_1 = 1/3, C_2 = -4/3 \quad \Rightarrow y(t) = \frac{1}{3}e^{2t} - \frac{4}{3}e^{t}$$