Math 3331 Differential Equations

4.4 Harmonic Motion

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4.4 Harmonic Motion

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  - Pendulum For Small ϕ
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- Classification of Harmonic Motion
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Harmonic Motion: Mass-Spring System

$\leftrightarrow$ "Homogeneous Equations"

Mass–spring system:

$$my'' + \mu y' + ky = 0$$

no forcing term

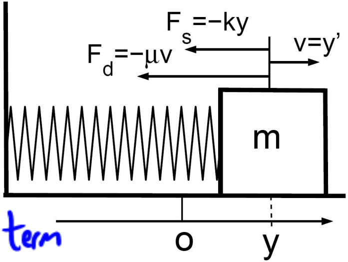
m: mass (kg)

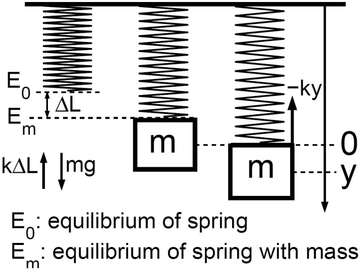
μ: damping constant (kg/s)

k: spring constant (kg/s^2)

y: deviation of mass position from equilibrium position (m)

y': velocity (m/s)





Harmonic Motion: Pendulum For Small ϕ

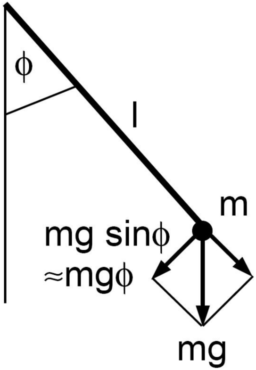
Pendulum for small ϕ:

$$\phi'' + (\mu/m)\phi' + (g/l)\phi = 0$$

ϕ: angle (no unit)

g: $9.8\, m/s^2$

l: length (m)

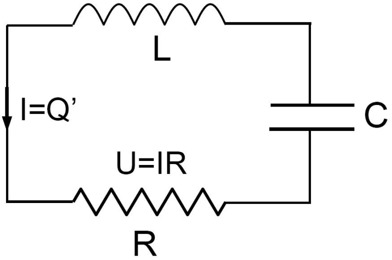


Harmonic Motion: RLC-Circuit

RLC–circuit:

$$LQ'' + RQ' + Q/C = 0$$

Q: charge at capacitor



- Q: charge (C)
- I: current Q' (A)
- L: inductivity (H)
- R: resistor (Ω)
- C: capacity (F)

Classification of Harmonic Motion: Mass-Spring System

Mass-Spring System

DE:

$$y'' + \frac{\mu}{m}y' + \frac{k}{m}y = 0,$$

$\rightarrow$ $m > 0, \quad k > 0, \quad \mu \geq 0$

Characteristic Eqn:

$$\lambda^2 + \underbrace{\frac{\mu}{m}}_{2c}\lambda + \underbrace{\frac{k}{m}}_{w_0^2} = 0$$

Roots:

$$\lambda_{1,2} = -\frac{\mu}{2m} \pm \frac{1}{2m}\sqrt{\mu^2 - 4km}$$

Classification

1. Undamped Case: No damping force

$$\mu = 0$$

2. Underdamped Case: Two small

$$0 < \mu^2 < 4km$$

3. Critically damped Case: getting critically bigger

$$\mu^2 = 4km$$

4. Overdamped Case: Quite big

$$\mu^2 > 4km$$

$$my'' + \mu y' + ky = F(t)$$

$$\hookrightarrow y'' + \frac{\mu}{m} y' + \underbrace{\frac{k}{m}}^{w_0^2} y = \frac{1}{m} F(t)$$

$$w_0 = \sqrt{\frac{k}{m}}$$

$$c = \frac{\mu}{2m} = \text{damping}$$

$$\boxed{x'' + 2c\, x' + w_0^2\, x = f(t)}$$

Harmonic Motion Equation

Mass-Spring System: Undamped Case ($\mu = 0$)

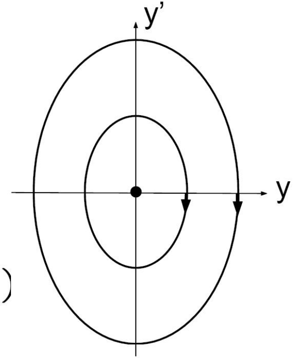
Undamped Case: $\mu = 0$

$$\lambda = \pm i\omega_0$$

$$\omega_0 = \sqrt{k/m}$$

general solution

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$



- oscillation – regular
- phase portrait: center
- clockwise direction of rotation

$\Rightarrow$ $x'' + w_0^2 x = 0 \Leftarrow$ no damping

Simple Harmonic Motion with no external force

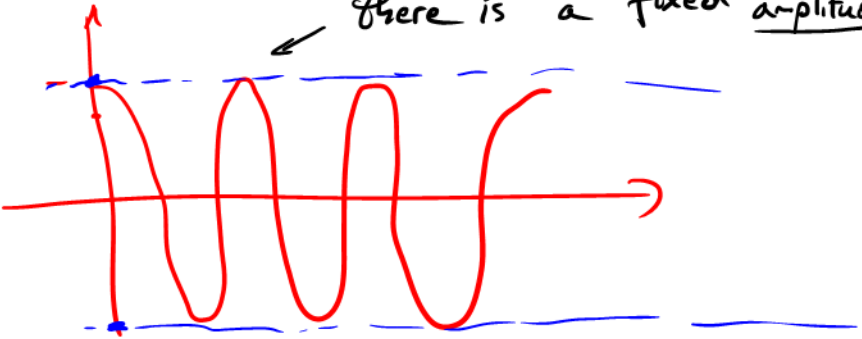
Homogeneous Eqn

$\lambda^2 + w_0^2 = 0$

$\lambda = \pm i w_0$

$$x(t) = \underbrace{a\cos(w_0 t) + b\sin(w_0 t)}_{\text{periodic function}}$$

there is a fixed "amplitude"



$$x(t+T) = x(t)$$

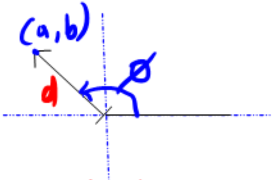
$$\boxed{T = \frac{2\pi}{\omega_0}} \leftarrow \text{period for } x(t).$$

- Undamped Harmonic Motion with no external force!

$\Rightarrow$ a fixed amplitude $= A$

Find it $x(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t)$

$$d = \sqrt{a^2+b^2} = \underset{\text{amplitude}}{A}$$



Calculus III

$$\begin{cases} a = A\cos\phi \\ b = A\sin\phi \end{cases}$$

$$\left\{ \phi = \tan^{-1}\left(\frac{b}{a}\right) \right.$$

Rewrite

$$\longrightarrow x(t) = \underbrace{A\cos\phi}_{a}\cos(w_0 t) + \underbrace{A\sin\phi}_{b}\sin w_0 t$$

$$x(t) = \underbrace{A}\cdot\cos(w_0 t - \phi)$$

amplitude

$$A = \sqrt{a^2 + b^2}$$

$$\phi = \tan^{-1}\left(\frac{b}{a}\right)$$

phase shift

Now, the motion is better understood.

Mass-Spring System: Underdamped Case ($0 < \mu^2 < 4km$)

• $y'' + 2c\,y' + \omega_0^2\,y = 0 \iff \lambda^2 + 2c\lambda + \omega_0^2 = 0$

I. **Underdamped Case:** $0 < \mu^2 < 4km$ $\lambda = -c \pm \sqrt{c^2 - \omega_0^2}$

$c < \omega_0$

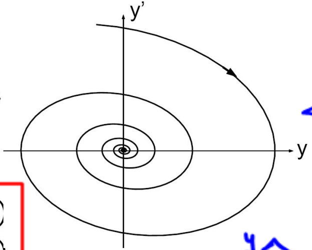
$$\lambda_{1,2} = -\alpha \pm i\omega$$

$$\alpha = \mu/2m$$

$$\omega = \sqrt{4km - \mu^2}/(2m$$

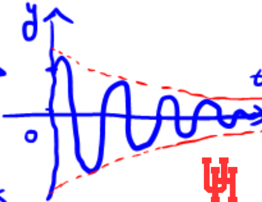
$$= \sqrt{\omega_0^2 - \mu^2/4m^2}$$

$$\boxed{y(t) = e^{-\alpha t}(c_1 \cos(\omega t) + c_2 \sin(\omega t))}$$



phase plane

- damped oscillation
- phase portrait: spiral sink
- clockwise direction of rotation



Still oscillating, but amplitude decreases

Harmonic Damped Motion

$$x'' + 2c\,x' + \omega_0^2 x = 0$$

$$\hookrightarrow \lambda^2 + 2c\,\lambda + \omega_0^2 = 0$$

$$\lambda = \frac{-2c \pm \sqrt{4c^2 - 4\omega_0^2}}{2}$$

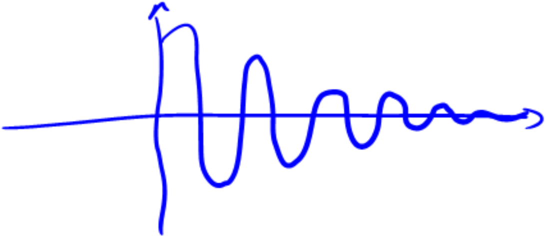
$$= -c \pm \sqrt{c^2 - \omega_0^2}$$

$$\lambda = -c \pm \sqrt{c^2 - w_0^2}$$

I. $c < w_0$

$$\Rightarrow \lambda = -c \pm i w$$

$$\Rightarrow x(t) = e^{-ct}\left(c_1 \cos wt + c_2 \sin wt\right)$$



Mass-Spring System: Critically damped Case ($\mu^2 = 4km$)

$y'' + 2cy' + \omega_0^2 y = 0 \Leftrightarrow \lambda^2 + 2c\lambda + \omega_0^2 = 0$

$\lambda = -c \pm \sqrt{\underbrace{c^2 - \omega_0^2}_{=0}} = -c$

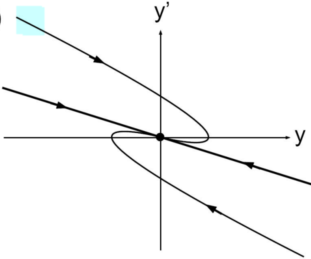
II. $c = \omega_0$

Critically Damped Case: $\mu^2 = 4km$

$-c = \lambda_1 = \lambda_2 = -\mu/(2m)$

$y(t) = e^{\lambda_1 t}(c_1 + c_2 t)$

- phase portrait: degenerate nodal sink



$\leftarrow$ phase plane

$\hookrightarrow y(t) = e^{-ct}(c_1 + c_2 t)$

II $c = \omega_0$ $\lambda = -c \pm \sqrt{\underbrace{c^2 - \omega_0^2}_{0}}$

$\Rightarrow \underline{\lambda = -c}$

$$x(t) = c_1 e^{-ct} + c_2 t \cdot e^{-ct}$$

$\Rightarrow$ oscillating disappears very fast.

Mass-Spring System: Overdamped Case ($\mu^2 > 4km$)

$y'' + 2cy' + \omega_0^2 y = 0 \iff \lambda^2 + 2c\lambda + \omega_0^2 = 0$

$\lambda_{1,2} = -c \pm \underbrace{\sqrt{c^2 - \omega_0^2}}_{\text{positive}}$

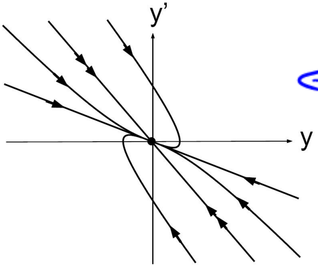
III. $c > \omega_0$

Overdamped Case: $\mu^2 > 4km$

$$\lambda_1 < \lambda_2 < 0$$

$\Rightarrow$ $\boxed{y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$

- phase portrait: nodal sink
- both eigenlines: negative slopes



$\leftarrow$ phase plane

III. $c > \omega_0$

$$\lambda = -c \pm \sqrt{\underbrace{c^2 - \omega_0^2}_{\text{positive}}}$$

λ_1 , λ_2 real

$$\Rightarrow \quad x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

overdamped

Harmonic Motion: Undamped Case

To Summarize:

Undamped Case:

$$\begin{aligned} y(t) &= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \\ &= A[d_1 \cos(\omega_0 t) + d_2 \sin(\omega_0 t)] \end{aligned}$$

where $\begin{cases} A = \sqrt{c_1^2 + c_2^2} \\ d_1 = c_1/A, \ d_2 = c_2/A \end{cases}$

Since $d_1^2 + d_2^2 = 1$ we can define ϕ by

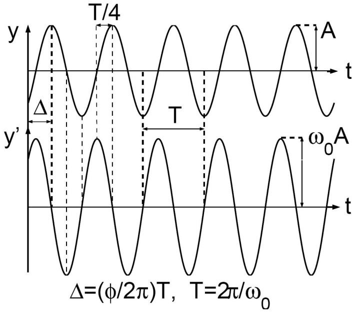
$$\begin{aligned} d_1 &= \cos\phi, \quad d_2 = \sin\phi \\ \Rightarrow \quad d_2/d_1 &= c_2/c_1 = \tan\phi \end{aligned}$$

$$\begin{aligned} y(t) &= A[\cos\phi \cos(\omega_0 t) + \sin\phi \sin(\omega_0 t)] \\ &= A\cos(\omega_0 t - \phi) \\ y'(t) &= -\omega_0 A \sin(\omega_0 t - \phi) \end{aligned}$$

A: **amplitude**

ϕ: **phase angle**, choose $-\pi < \phi \le \pi$

$$\phi = \begin{cases} \arctan(c_2/c_1) & \text{if } c_1 > 0 \\ \arctan(c_2/c_1) + \pi & \text{if } c_1 < 0, c_2 \ge 0 \\ \arctan(c_2/c_1) - \pi & \text{if } c_1 < 0, c_2 < 0 \\ \pi/2 & \text{if } c_1 = 0, c_2 > 0 \\ -\pi/2 & \text{if } c_1 = 0, c_2 < 0 \end{cases}$$



Harmonic Motion: Underdamped Case

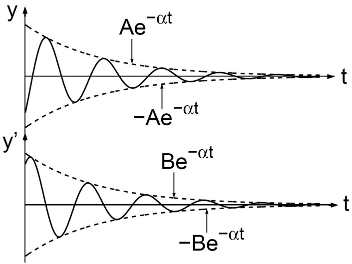
I. $c < \omega_0$ Underdamped

↙ picture

Underdamped Case:

$$\begin{aligned} y(t) &= e^{-\alpha t}[c_1 \cos(\omega t) + c_2 \sin(\omega t)] \\ &= e^{-\alpha t} A \cos(\omega t - \phi) \\ y'(t) &= e^{-\alpha t}[(\omega c_2 - \alpha c_1)\cos(\omega t) \\ &\qquad -(\omega c_1 + \alpha c_2)\sin(\omega t)] \\ &= e^{-\alpha t} B \cos(\omega t - \psi) \end{aligned}$$

$\pm A e^{-\alpha t}$, $\pm B e^{-\alpha t}$: envelopes of damped oscillations



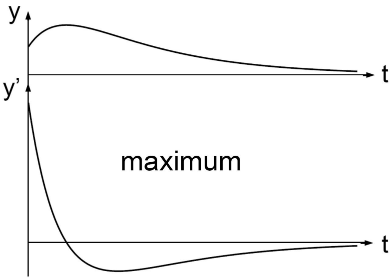
Harmonic Motion: Critically and Overdamped Cases

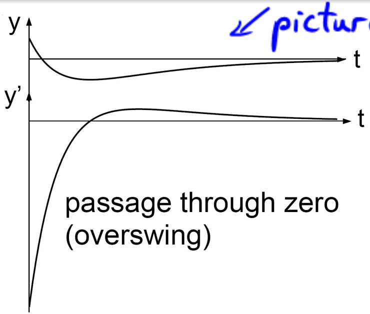
II. $c = \omega_0$

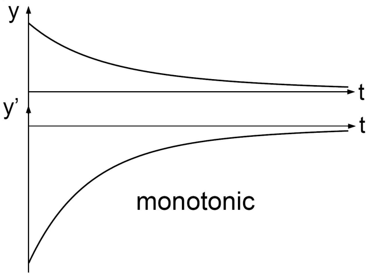
Critically and Overdamped Cases:

↙ picture

- If $y(0)$ and $y'(0)$ have equal signs, then
  - $y(t)$ attains maximum or minimum
  - $y'(t)$ crosses zero
- If $y(0)$ and $y'(0)$ have opposite signs, then $y(t)$
  - crosses zero if $|y'(0)/y(0)|$ is large
  - is monotonic if $|y'(0)/y(0)|$ is small







Exercise 4.4.11

exercise

equation

no damping

$$\boxed{0.2\,y'' + 5\,y = 0}$$

Ex. 4.4.11: Given an undamped mass-spring system with $m = 0.2\,kg$, $k = 5\,kg/s^2$, $y(0) = 0.5\,m$, $y'(0) = 0$, find amplitude, frequency, phase of motion.

Natural frequency: $\omega_0 = \sqrt{5/0.2} = 5/s \Rightarrow$

$$y(t) = c_1 \cos 5t + c_2 \sin 5t, \quad y'(t) = -5c_1 \sin 5t + 5c_2 \cos 5t$$

$$\text{IC: } y(0) = c_1 = 0.5, \; y'(0) = 5c_2 = 0 \;\Rightarrow\; y(t) = 0.5 \cos 5t$$

$$\Rightarrow \text{ amplitude: } A = 0.5\,m, \quad \text{phase: } \phi = 0$$

Solution

$$y'' + \frac{5}{0.2}\,y = 0 \iff y'' + \underbrace{25}_{\omega_0^2}\,y = 0$$

$\omega_0^2 \to \omega_0 = 5$

$$A = \sqrt{c_1^2 + c_2^2}, \quad \phi = \tan\left(\frac{c_2}{c_1}\right)$$

Exercise 4.4.22

Ex. 4.4.22: A mass-spring system with $m = 0.1\,kg$, $k = 9.8\,kg/s^2$ is placed in a viscous medium with friction force $0.3\,N$ if $v = 0.2\,m/s$. Initial data: $y(0) = 0.1\,m$, $y'(0) = 0$. Find amplitude, frequency, and phase of motion.

Friction coefficient: $F_d = \mu v \Rightarrow 0.3 = \mu 0.2 \Rightarrow \mu = 1.5\,kg/s$.

$$\text{ODE:}\quad 0.1y'' + 1.5y' + 9.8y = 0 \;\Rightarrow\; y'' + 15y' + 98y = 0 \;\Rightarrow$$

$p(\lambda) = \lambda^2 + 15\lambda + 98 = (\lambda + 7.5)^2 + 41.75 \;\Rightarrow\; \lambda = -7.5 \pm i\sqrt{41.75} \approx -7.5 \pm 6.461i$

$\Rightarrow$ Damped motion with frequency $\omega \approx 6.461\,/s$ of harmonic part $\Rightarrow$

$$y(t) = e^{-7.5t}(c_1 \cos \omega t + c_2 \sin \omega t)$$

$$y'(t) = e^{-7.5t}[(\omega c_2 - 7.5c_1)\cos \omega t - (\omega c_1 + 7.5c_2)\sin \omega t]$$

$$\text{IC:}\;\; y(0) = c_1 = 0.1,\;\; y'(0) = \omega c_2 - 7.5c_1 = 0 \;\;\Rightarrow\;\; c_2 = 7.5c_1/\omega \approx 0.116$$

$$\Rightarrow\; y(t) = e^{-7.5t}(0.1 \cos \omega t + 0.116 \sin \omega t)$$

Exercise 4.4.22 (cont.)

Amplitude of harmonic part: $A_0 \approx \sqrt{0.1^2 + 0.116^2} \approx 0.153$. Since $c_1, c_2 > 0$
$\Rightarrow$ phase angle $\phi = \arctan(c_2/c_1) \approx \arctan(1.16) \approx 0.859$

$$\Rightarrow y(t) = 0.153e^{-7.5t}\cos(6.461t - 0.859)$$

$\Rightarrow$ amplitude: $A(t) = 0.153e^{-7.5t}\,m$,
frequency: $\omega = 6.461/s$, phase: $\phi = 0.859$

ex spring, $\mu = 12.8\ kg/s$, $m = 4\,kg$, $k = 169$

- Find the solution equation,
  given that $y(0) = 0.1$, $y'(0) = 1.3\ m/s$

Solution: Write DE:

$$\boxed{4 \cdot y'' + 12.8\, y' + 169 \cdot y = 0}$$

just solve

$$y'' + 3.2y' + 42.25y = 0$$

$$\lambda^2 + 3.2\lambda + 42.25 = 0$$

$$\lambda_{1,2} = -1.6 \pm i\,6.3$$

$$y(t) = e^{-1.6t}\left(c_1\cos(6.3t) + c_2\sin(6.3t)\right)$$

$$y(0) = 0.1, \quad y'(0) = 1.3$$

$C_1 = 0.1$

$C_2 = 0.23$

$$x(t) = e^{-1.6t}\left(0.1 \cdot \cos(6.3t) + 0.23\sin(6.3t)\right)$$