Math 3331 Differential Equations

4.5 Inhomogeneous Equations

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4.5 Inhomogeneous Equations

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Inhomogeneous Equations and General Solution

General Solution to Inhomogeneous Equation

The general solution to the inhomogeneous linear equation

$$y'' + p(t)y' + q(t)y = f(t)$$

is given by

$$y(t) = y_p(t) + \boxed{C_1 y_1(t) + C_2 y_2(t)}$$

homogeneous solution

where C_1 and C_2 are arbitrary constants, and

- $y_p(t)$ is a particular solution to the inhomogeneous equation,
- $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions to the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

Method of Undetermined Coefficients

How to find the particular solution?

The method of undetermined coefficients is based on the fact that there are some situations where the form of the forcing term in DE allows us to almost guess the form of a particular solution.

Key Idea

If the forcing term $f(t)$ has a form that is replicated under differentiation, then look for a solution with the same general form as the forcing term.

example: $f(t) = e^{at}$

Try: $y_p(t) = C \cdot e^{at}$

Exponential Forcing Terms

Try $y_p = a \cdot e^{-t}$

Ex.: $y'' + y = e^{-t}$

- **Trial Form:** $y_p(t) = ae^{-t}$
- Sub y_p in ODE $\Rightarrow$

$$y_p'' + y_p = ae^{-t} + ae^{-t} = 2ae^{-t} \stackrel{!}{=} e^{-t}$$

$\Rightarrow 2a = 1 \Rightarrow y_p(t) = e^{-t}/2$ is P.S.

$a \cdot \cancel{e^{-t}} + a \cdot \cancel{e^{-t}} = \cancel{e^{-t}} \Rightarrow 2a = 1$

$a = 1/2$

$y_p(t) = \frac{1}{2} e^{-t}$ ✓

Homogeneous Eqn: $y'' + y = 0$ $\qquad \lambda = a + bi$

$\hookrightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = i$

$= \underset{a}{0} + \underset{b}{1} \cdot i$

$b = 1$

$y_1(t) = e^{at} \cos(bt) = \cos t$

$y_2(t) = e^{at} \sin(bt) = \sin t$

$\Rightarrow y(t) = \frac{1}{2} e^{-t} + C_1 \cos t + C_2 \sin t$

Exponential Forcing Terms

Ex.: $y'' + y = te^{-t}$ $\Rightarrow$ $y_p(t) = a \cdot te^{-t}$

Try: sub $y_p(t) = ate^{-t}$ in ODE

$$\Rightarrow \; a(-2e^{-t} + te^{-t}) \stackrel{?}{=} te^{-t}$$

Doesn't work $\longrightarrow$ use $y_p(t) = (a+bt)e^{-t}$

$$y_p'' + y_p = [a + b(-2+t)]e^{-t} + (a+bt)e^{-t}$$

$$= [(2a-2b) + 2bt]e^{-t} \stackrel{!}{=} te^{-t}$$

$$\Rightarrow \left\{ \begin{array}{rcl} 2a - 2b & = & 0 \\ 2b & = & 1 \end{array} \right\} \Rightarrow a = b = 1/2$$

$\Rightarrow y_p(t) = (1+t)e^{-t}/2$ is P.S.

Exponential Forcing Terms

Particular Solution

• Try: $y_p = a e^{-t}$ ✗

$a e^{-t} - a e^{-t} = e^{-t}$

$0 \neq e^{-t}$

Ex.: $y'' - y = e^{-t}$

Try: sub $y_p(t) = ae^{-t}$ in ODE

$$\Rightarrow \; y_p'' - y_p = 0 \stackrel{?}{=} e^{-t}$$

special

Doesn't work $\longrightarrow$ use $y_p(t) = ate^{-t}$ linear

$$y_p'' - y_p = a(-2+t)e^{-t} - a(te^{-t})$$

$$= -2ae^{-t} \stackrel{!}{=} e^{-t}$$

$\Rightarrow a = -1/2 \Rightarrow y_p(t) = -te^{-t}/2$ is P.S.

(12)

$\rightarrow$ Try $\Rightarrow$ $y_p = \underline{(at+b)} \cdot e^{-t}$

$y_p' = a \cdot e^{-t} - ate^{-t} - be^{-t}$

$y_p'' = -ae^{-t} - a \cdot e^{-t} + ate^{-t} + be^{-t}$

$\hookrightarrow y'' - y = e^{-t}$

$(-2a + \cancel{at} + \cancel{b}) - (\cancel{at} + \cancel{b}) = 1$

$-2a = 1 \Longrightarrow a = -\frac{1}{2}$

Hom. solution

$$y'' - y = 0$$

$$\lambda^2 - 1 = 0$$

$$(\lambda - 1)(\lambda + 1) = 0$$

$$\lambda = -1 \implies y_1(t) = e^{-t}$$

Trig Forcing Terms

Ex.: $y'' + y' + 2y = \cos t$

Try: sub $y_p = a\cos t$ in ODE $\Rightarrow$

$$\begin{aligned} & a(-\cos t - \sin t + 2\cos t) \\ = \quad & a(\cos t - \sin t) \stackrel{?}{=} \cos t \end{aligned}$$

Doesn't work!
We need $\cos t$ *and* $\sin t$:

$$y_p(t) = a\cos t + b\sin t$$

$$\begin{aligned} & \Rightarrow y_p'' + y_p' + 2y_p \\ & = (a+b)\cos t + (-a+b)\sin t \stackrel{!}{=} \cos t \end{aligned}$$

$$\Rightarrow \left\{ \begin{array}{rcl} a+b & = & 1 \\ -a+b & = & 0 \end{array} \right\} \Rightarrow a = b = 1/2$$

$\Rightarrow y_p(t) = (\cos t + \sin t)/2$ is P.S.

$$y'' + y' + 2y = \cos t \quad (*)$$

Try $\quad y_p = a\cdot\cos t + b\sin t$

$$y_p' = -a\sin t + b\cos t$$

$$y_p'' = -a\cos t - b\sin t$$

$$(-a\cos t - b\sin t) - a\sin t) + b\cos t)$$

$$+2a\cos t + 2b\sin t = \cos t$$

$$\underbrace{(a+b)}_{1}\cos t + \underbrace{(b-a)}_{0}\sin t = \cos t$$

$$a = b = \frac{1}{2}$$

$$y_p = \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

$\Longrightarrow$ Homogeneous Solutions:

$$\lambda^2 + \lambda + 2 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2}}{2}$$

$$= \frac{-1}{2} \pm i\,\frac{\sqrt{7}}{2}.$$

$$\lambda = \frac{1}{2} + i\,\frac{\sqrt{7}}{2}$$

$$y_1(t) = e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right)$$

$$y_2(t) = e^{\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right)$$

$$\Rightarrow\; y(t) = \frac{1}{2}\cos t + \frac{1}{2}\sin t + c_1 y_1 + c_2 y_2.$$