Math 3331 Differential Equations

4.6 Variation of Parameters

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4.6 Variation of Parameters

- Second-Order Equations and Planar Systems
- Fundamental Set of Solutions and Fundamental Matrix
- Variation of Parameters for Systems
- Particular Solution to System

Second-Order Equations and Planar Systems

Inhomogeneous equation:

$$y'' + a(t)y' + b(t)y = F(t) \quad (8)$$

Homogeneous equation:

$$y'' + a(t)y' + b(t)y = 0 \quad (9)$$

Inhomogeneous system:

$$\mathbf{x}' = A(t)\mathbf{x} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \quad (10)$$

where $A(t) = \begin{bmatrix} 0 & 1 \\ -b(t) & -a(t) \end{bmatrix}$

$$\Rightarrow y'' + a(t)\,y' + b(t)\,y = F(t)$$

$$\Rightarrow \quad x = y' \qquad \Longleftrightarrow \qquad y' = x$$

$$y'' = x' \qquad\qquad\qquad x' = y''$$

$$\begin{cases} y' = x = \begin{bmatrix} -0\cdot y + 1\cdot x \\ -b(t)\,y - a(t)\,x \end{bmatrix} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \\ x' = y'' = \end{cases}$$

$$\underbrace{\begin{bmatrix} y' \\ x' \end{bmatrix}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -b(t) & -a(t) \end{bmatrix}}_{A(t)} \underbrace{\begin{bmatrix} y \\ x \end{bmatrix}}_{\vec{x}} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

$$\vec{x}\,' = A(t) \cdot \vec{x} + \begin{bmatrix} 0 \\ F(t) \end{bmatrix}$$

Solve this matrix equation for $\vec{x}$.

Fundamental Set of Solutions and Fundamental Matrix

Homogeneous solutions

Let $y_1(t), y_2(t)$ be F.S.S for (9)

$$\Rightarrow X(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$$ = Wronskian matrix

is F.M. of $\mathbf{x}' = A(t)\mathbf{x}$, and

$$X(t)^{-1} = \frac{1}{W(t)} \begin{bmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{bmatrix}$$

where $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$.

A matrix-valued function $X(t)$ is a fundamental matrix for the system $x' = A(t)x$ if and only if

$$X' = A(t)X, \quad X_0 = X(t_0) \text{ is invertible for some } t_0$$

planar system

Variation of Parameters for Systems

Look for a solution of the form

particular solution $$x_p(t) = X(t)\underline{v(t)}, \quad v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$$

Differentiating it gives

$$x_p' = X'v + Xv' = A(t)Xv + Xv' = A(t)x_p + Xv'$$

x_p is to be a solution to $x_p' = A(t)x_p + f(t)$ provided that

$$Xv' = f(t) \quad \Rightarrow \quad v' = X^{-1}f(t) \quad \Rightarrow \quad v(t) = \int X^{-1}(t)f(t)dt$$

Inserting v gives the particular solution

$$x_p(t) = X(t)\left(\int X^{-1}(t)f(t)dt\right)$$

Variation of Parameters for Systems (cont.)

$$\text{Set} \quad \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \int X(t)^{-1} \begin{bmatrix} 0 \\ F(t) \end{bmatrix} dt$$

$$\Rightarrow \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \int \begin{bmatrix} -y_2(t) \\ y_1(t) \end{bmatrix} \frac{F(t)}{W(t)} dt$$

this is what this method is about

$$\boxed{\begin{aligned} v_1(t) &= -\int y_2(t)\frac{F(t)}{W(t)}dt \\ v_2(t) &= \int y_1(t)\frac{F(t)}{W(t)}dt \end{aligned}}$$

← Remember!

(recall that $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$)

Particular Solution to System

Particular solution of (10):

$$\mathbf{x}_p(t) = X(t)\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t)v_1(t) + y_2(t)v_2(t) \\ y_1'(t)v_1(t) + y_2'(t)v_2(t) \end{bmatrix}$$

$y_p(t)$

First component is part. sol. of (8):

$$y_p(t) = y_1(t)v_1(t) + y_2(t)v_2(t)$$

where $\begin{cases} v_1(t) = -\int [y_2(t)F(t)/W(t)]\,dt \\ v_2(t) = \int [y_1(t)F(t)/W(t)]\,dt \end{cases}$

Example

Ex.: Find particular solution of

$$y'' + y = \tan t$$

F.S.S. of $y'' + y = 0$:

H.S. $\left\{\begin{array}{l} y_1(t) = \cos t \\ y_2(t) = \sin t \end{array}\right\} W(t) = \cos^2 t + \sin^2 t = 1$

P.S.

$$\begin{aligned} v_1(t) &= -\int \sin t \tan t \, dt \\ &= \sin t - \ln|\sec t + \tan t| \\ v_2(t) &= \int \cos t \tan t \, dt = \cos t \end{aligned}$$

$\Rightarrow y_p(t) = -\cos t \ln|\sec t + \tan t| = v_1 \cos t + v_2 \cdot \sin t$

simplify

$\Rightarrow y(t) = C_1 \cos t + C_2 \sin t - \cos t \ln(\sec t + \tan t)$

$y'' + y = \tan t \Rightarrow$ Find the Particular Solution.

$\rightarrow$ Homogeneous Solution

$y'' + y = 0 \Rightarrow \lambda^2 + 1 = 0$

$\lambda = \underline{\underline{i}} = \underbrace{0}_{a} + \underbrace{i}_{b=1}$

$\Rightarrow \begin{cases} y_1 = e^{at}\cos(bt) = \cos t \\ y_2 = \sin t \end{cases}$

$\underline{\underline{\text{H.S.}}}$

Particular Solution: Variation of Parameters

$y_H = c_1 y_1 + c_2 \cdot y_2$ $\qquad$ $y_p = v_1 \cdot y_1 + v_2 \cdot y_2$

Our goal

Need to find v_1 & v_2.

- To find v_1 and v_2, always we need two constraint equations.

$\rightarrow$ One comes easily from $y'' + y = \tan t$

- Find a new constraint equation:

$$y_p = v_1 \cdot \cos t + v_2 \sin t$$

$$\hookrightarrow y_p' = v_1' \cos t - v_1 \sin t + v_2' \cdot \sin t + v_2 \cos t$$

$$= \underbrace{\left(v_1' \cos t + v_2' \sin t \right)}_{=0} - \underline{v_1 \sin t + v_2 \cos t}$$

Not nice

make it zero

$$\Rightarrow \boxed{v_1' \cos t + v_2' \sin t = 0} \; 2$$

$$y_p'' = -v_1' \sin t - v_1 \cos t + v_2' \cos t - v_2 \sin t$$

$$= -v_1' \sin t + v_2' \cos t - v_1 \cos t - v_2 \sin t$$

$$y_p' = -v_1 \sin t + v_2 \cos t$$

- One equation comes from

$$y'' + y = \tan t \quad (*)$$

Substitute y_p'', y_p in $(*)$

$$y_p'' + y_p = \boxed{-v_1' \sin t + v_2' \cos t = \tan t}$$

Simplified version

previous slide

$v_1 \cos t + v_2 \sin t$

Two Equations

Find solution

$$\begin{cases} v_1' \cos t + v_2' \sin t = 0 \\ -v_1' \sin t + v_2' \cos t = \tan t \end{cases}$$

Matrix Form

$$\xrightarrow[\text{reveals the method}]{\text{Matrix}} \underbrace{\begin{bmatrix} \underbrace{\cos t}_{y_1} & \underbrace{\sin t}_{y_2} \\ \underbrace{-\sin t}_{y_1'} & \underbrace{\cos t}_{y_2'} \end{bmatrix}}_{\text{Wronskian of } y_1 \text{ and } y_2.} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \underbrace{\tan t}_{f(t)} \end{bmatrix}$$

$\Rightarrow$ It has a solution

$$\begin{array}{l} \sin t \\ + \\ \cos t \end{array} \begin{cases} v_1' \cos t + \overset{\sin t}{v_2'} \sin t = 0 \cdot \sin t \\ -v_1' \sin t + \overset{\cos t}{v_2'} \cos t = \tan t \cdot \cos t \end{cases}$$

$$v_2' \left(\sin^2 t + \cos^2 t \right) = \sin t$$

$$\Rightarrow v_2' = \frac{\overset{y_1}{\cos t} \cdot \overset{f(t)}{\tan t}}{\underbrace{\sin t \cdot \sin t + \cos t \cdot \cos t}_{W(t)}}$$

$$v_2' = \frac{y_1 \cdot f(t)}{W(t)}$$

$$\Rightarrow \boxed{\begin{aligned} v_2 &= \int \frac{y_1 f(t)}{W(t)}\, dt \\ v_1 &= -\int \frac{y_2 \cdot f(t)}{W(t)}\, dt \end{aligned}}$$

Done