# Math 3331 Differential Equations

## 4.7 Forced Harmonic Motion

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# 4.7 Forced Harmonic Motion

- Periodically Forced Harmonic Motion
- Forced Undamped Harmonic Motion: Beats
- Forced Undamped Harmonic Motion: Resonance
- Forced Damped Harmonic Motion
- Amplitude and Phase

# Periodically Forced Harmonic Motion

– Recall Undetermined Coefficient Method

Periodically forced mass-spring system: $mx'' + \mu x' + kx = F_0 \cos \omega t$

or $x'' + dx' + \omega_0^2 x = A \cos \omega t$

where $d = \mu/m$, $\omega_0 = \sqrt{k/m}$, $A = F_0/m$

external force, sinusoidal force

*Sinusoidal forcing*: $F(t) = A \cos \omega t$

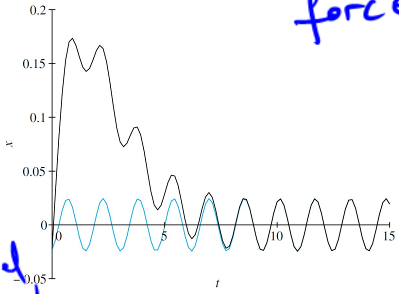
where $A$ is the amplitude and $\omega$ is the driving frequency.

*General solution*:

$$x(t) = x_h(t) + x_p(t)$$

where

- $x_p(t)$: steady state part (persistent oscillation) Coming from external force!
- $x_h(t)$: transient part ($d > 0$) ($x_h(t) \to 0$ for $t \to \infty$)



# Forced Undamped Harmonic Motion: Beats ($\omega \neq \omega_0$)

No damping force

$$x'' + \omega_0^2 x = A\cos\omega t \qquad (1)$$

Try particular solution:

$$x_p(t) = a\cos\omega t + b\sin\omega t \Rightarrow$$

$$x_p'' + \omega_0^2 x_p = (\omega_0^2 - \omega^2)(a\cos\omega t + b\sin\omega t)$$

The r.h.s. is equal to $A\cos\omega t$ if

$$(\omega_0^2 - \omega^2)a = A, \quad (\omega_0^2 - \omega^2)b = 0$$

$\Rightarrow a = A/(\omega_0^2 - \omega^2)$, $b = 0 \Rightarrow$

$$x_p(t) = [A/(\omega_0^2 - \omega^2)]\cos\omega t \qquad (2)$$

To find general solution, add general solution of

$$x'' + \omega_0^2 x = 0 \qquad (3)$$

F.S.S. for (3): $\cos\omega_0 t, \sin\omega_0 t$

$\Rightarrow$ general solution of (1):

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + x_p(t)$$

Solution

H.S.

$x_h = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$

Method of Undetermined Coefficients

# Beats: $\omega \neq \omega_0$

Case I : $\omega \neq \omega_0$

**Beats.** Assume IC: $x(0) = 0, x'(0) = 0$

$$\Rightarrow \; c_1 = -A/(\omega_0^2 - \omega^2), \; c_2 = 0 \; \Rightarrow$$

$$x(t) = \frac{A}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t) \quad (4)$$

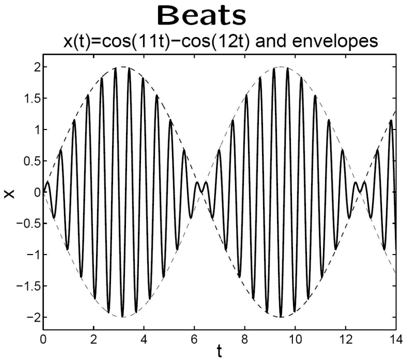
Set $\delta = (\omega_0 - \omega)/2, \; \overline{\omega} = (\omega_0 + \omega)/2$
Use $(\alpha = \omega t, \; \beta = \omega_0 t)$

$$\cos\alpha - \cos\beta = 2\sin\left(\frac{\beta - \alpha}{2}\right)\sin\left(\frac{\beta + \alpha}{2}\right)$$

$$\Rightarrow \; x(t) = \frac{A\sin\delta t}{2\overline{\omega}\delta}\sin\overline{\omega}t \qquad (5)$$

If $\delta << \overline{\omega} \Rightarrow [A/(2\overline{\omega}\delta)]\sin\delta t$ is slowly varying envelope

# Beats in Forced, Undamped, Harmonic Motion




$w=11$, $w_0=12$
slightly different

In acoustics, a beat is an interference between two sounds of slightly different frequencies, perceived as periodic variations in volume whose rate is the difference between the two frequencies.

$$x(t) = \cos wt - \cos w_0 t = 2 \sin \delta t \sin \bar{\omega} t$$

where the mean frequency $\bar{\omega}$ and the half difference $\delta$ are defined by

$$\bar{\omega} = (\omega_0 + \omega)/2, \quad \delta = (\omega_0 - \omega)/2.$$

# Forced Undamped Harmonic Motion: Resonance ($\omega = \omega_0$)

Case II: $\omega = \omega_0$ Frequency of exterior force reaches the natural frequency of the system.

**Resonant Case:** $\omega = \omega_0$

Solution (2) is not valid if $\omega = \omega_0$. In this case try

$$x_p(t) = t(a\cos\omega_0 t + b\sin\omega_0 t)$$

$$\Rightarrow x_p'' + \omega_0^2 x_p = -2a\sin\omega_0 t + 2b\cos\omega_0 t$$

The r.h.s. equals $A\cos\omega_0 t$ if

$$a = 0, 2\omega_0 b = A \Rightarrow b = A/(2\omega_0)$$

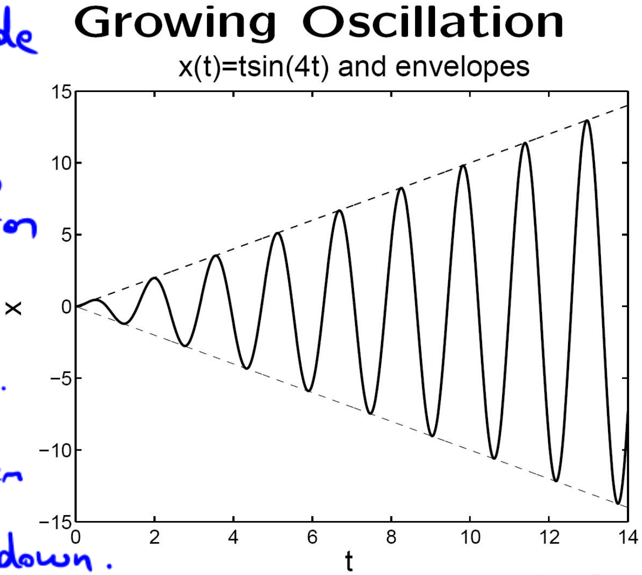
$\Rightarrow$ $x_p(t) = [A/(2\omega_0)]t\sin\omega_0 t$ (linearly growing oscillation)

*Note:* $x_p(0) = 0$, $x_p'(0) = 0$. **Ex.:** $A = 8$, $\omega_0 = 4 \Rightarrow x_p(t) = t\sin 4t$

it grows as t grows

# Resonance in Forced, Undamped, Harmonic Motion

Amplitude of oscillation keeps growing. The system breaks down.



# Forced Damped Harmonic Motion

Complex Method — Skip it!

$$x'' + dx' + \omega_0^2 x = A\cos\omega t \qquad (6)$$

Since $A\cos\omega t = \text{Re}(Ae^{i\omega t})$, any solution $x(t)$ is the real part of a solution $z(t)$ of

$$z'' + dz' + \omega_0^2 z = Ae^{i\omega t} \qquad (7)$$

*Solution Strategy:*

- Find particular solution of (7)
- Real part $\rightarrow$ particular solution of (6)

# Particular Solution of (7)

Try **complex exponential** for (7):

$$z_p(t) = ae^{i\omega t} \Rightarrow z_p'' + dz_p' + \omega_0^2 z_p = ((i\omega)^2 + i\omega d + \omega_0^2)ae^{i\omega t} = Ae^{i\omega t}$$

$$\Rightarrow [(\omega_0^2 - \omega^2) + i\omega d]a = A$$

$$\Rightarrow \frac{a}{A} = \frac{1}{(\omega_0^2 - \omega^2) + i\omega d}$$

Use $1/(\alpha + i\beta) = (\alpha - i\beta)/(\alpha^2 + \beta^2)$

$$\Rightarrow \frac{a}{A} = \frac{(\omega_0^2 - \omega^2) - i\omega d}{D}$$

where $D = (\omega_0^2 - \omega^2)^2 + \omega^2 d^2$

# Amplitude and Phase

**Amplitude and Phase:** Set

$$a/A = Ge^{-i\phi} = G\cos\phi - iG\sin\phi$$

$$\Rightarrow G^2 = \left(\frac{(\omega_0^2-\omega^2)}{D}\right)^2 + \left(\frac{\omega d}{D}\right)^2$$

$$= \frac{(\omega_0^2-\omega^2)^2+\omega^2 d^2}{D^2} = \frac{D}{D^2}$$

$\Rightarrow G = 1/\sqrt{D} \equiv G(\omega)$ (*gain*), hence

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2 d^2}} \qquad (8)$$

*Phase angle:*

$$\omega_0^2-\omega^2 = G\cos\phi,\ \omega d = G\sin\phi$$

where $0 \le \phi < \pi$ (since $\sin\phi \ge 0$)

$$\Rightarrow \phi(\omega) = \text{arccot}\left(\frac{\omega_0^2-\omega^2}{\omega d}\right) \qquad (9)$$

# Solution of (6)

**Particular Solution of (6)**:

$$z_p(t) = ae^{i\omega t} = G(\omega)Ae^{i(\omega t-\phi)} \Rightarrow$$

$$x_p(t) = \text{Re} z_p(t) = GA\cos(\omega t - \phi) \quad (10)$$

**General Solution of (6)**:

$$x(t) = x_h(t) + x_p(t) \quad (11)$$

where $x_h(t) = c_1x_1(t) + c_2x_2(t)$ (12)

and $x_1(t)$, $x_2(t)$ is F.S.S. of

$$x'' + dx' + \omega_0^2 x = 0$$

**Steady State and Transient Parts:**

- $x_p(t)$: steady state part (persistent oscillation)
- $x_h(t)$: transient part ($d > 0$ $\Rightarrow x_h(t) \to 0$ for $t \to \infty$)

# Qualitative Forms

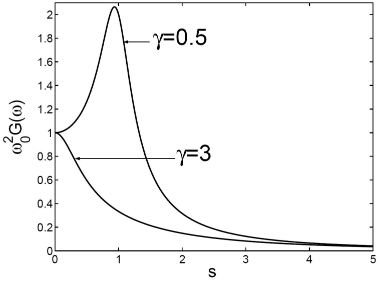
**Qualitative Forms of $G(\omega)$, $\phi(\omega)$:**

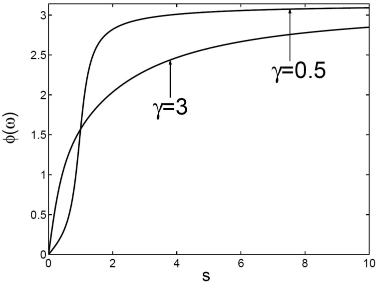
Set $s = \omega/\omega_0$, $\gamma = d/\omega_0 \Rightarrow$

$$\omega_0^2 G(\omega) = \frac{1}{\sqrt{(1-s^2)^2 + s^2\gamma^2}}$$

$$\phi(\omega) = \operatorname{arccot}\left(\frac{1-s^2}{s\gamma}\right)$$

- $G(\omega)$ has max at $s_m = \sqrt{1-\gamma^2/2}$, $\omega_0^2 G_m = 2/(\gamma\sqrt{4-\gamma^2})$, if $\gamma < \sqrt{2}$, and is monotonic for $\gamma > \sqrt{2}$
- $\phi(\omega)$ is "steep" for small $\gamma$ and "flat" for large $\gamma$ 3



# Example 4.7.18

**Ex.:** Consider a mass-spring system with $m = 5\,kg$, $\mu = 7\,kg/s$, $k = 3\,kg/s^2$, and a forcing term $2\cos 4t\ N$

(a) Find the steady periodic solution $x_p(t)$ and determine its amplitude and phase.

*Answer:* Equation: $5x'' + 7x' + 3x = 2\cos 4t \Rightarrow x'' + 1.4x' + 0.6x = 0.4\cos 4t$

Use complex method: $x_p(t) = \text{Re} z_p(t)$, where $z_p$ is particular solution of

$$z'' + 1.4z' + 0.6z = 0.4e^{4it}$$

Try $z_p = ae^{4it} \Rightarrow (-16 + 5.6i + 0.6)ae^{4it} = 0.4e^{4it}$

$$\Rightarrow a = \frac{0.4}{-15.4 + 5.6i} = \frac{0.4 \times (-15.4 - 5.6i)}{15.4^2 + 5.6^2} = -0.0229 - 0.0083i$$

$$\begin{aligned}\Rightarrow z_p(t) &= (-0.0229 - 0.0083i)(\cos 4t + i\sin 4t)\\ \Rightarrow x_p(t) &= \text{Re}(z_p(t)) = 0.0083\sin 4t - 0.0229\cos 4t \ \text{(superposition form)}\end{aligned}$$

To find amplitude and phase compute polar form: $a = A_0 e^{-i\phi}$, where

$$\begin{aligned}A_0 &= \sqrt{0.0229^2 + 0.0083^2} = 0.0244\\ \phi &= \text{arccot}(-0.0229/0.0083) = 2.7939\end{aligned}$$

$$\begin{aligned}\Rightarrow z_p(t) &= A_0 e^{i(4t-\phi)}\\ \Rightarrow x_p(t) &= 0.0244\cos(4t - 2.7939) \ \text{(amplitude-phase form)}\end{aligned}$$

# Example 4.7.18 (cont.)

(b) Find the position $x(t)$ if $x(0) = 0$, $x'(0) = 1\, m/s$

*Answer:* Find transient part: $x'' + 1.4x' + 0.6x = 0 \Rightarrow p(\lambda) = \lambda^2 + 1.4\lambda + 0.6 = 0$
$\Rightarrow \lambda = -0.7 \pm 0.3317i$

$\Rightarrow x_h(t) = e^{-0.7t}[c_1 \cos(0.3317t) + c_2 \sin(0.3317t)]$ and $x(t) = x_h(t) + x_p(t)$

Match $c_1, c_2$ to IC: (use superposition form)

$$\left.\begin{array}{rcl} x(0) & = & c_1 - 0.0229 = 0 \Rightarrow c_1 = 0.0229 \\ x'(0) & = & -0.7c_1 + 0.3317c_2 + 4 \times 0.0083 = 1 \Rightarrow c_2 = 2.9630 \end{array}\right\} \Rightarrow$$

$$x(t) = e^{-0.7t}[0.0229 \cos(0.3317t) + 2.9630 \sin(0.3317t)] + 0.0244 \cos(4t - 2.7939)$$