Math 3331 Differential Equations

5.3 The Inverse Laplace Transform

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5.3 The Inverse Laplace Transform

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Uniqueness Theorem

Thm.: If $f(t)$ and $g(t)$ are piecewise continuous on $0 \leq t < \infty$ and $\mathcal{L}(f)(s) = \mathcal{L}(g)(s)$ for $s > a$, then $f(t) = g(t)$ for all t in $0 \leq t < \infty$ at which $f(t)$ is continuous.

Proof: Let $f(t)$, $g(t)$ be piecewise continuous functions on $[0, \infty)$ and $\mathcal{L}(f)(s) = \mathcal{L}(g)(s)$.
Define $h(t) = f(t) - g(t)$. Note that

$h(t)$ is piecewise-continuous on $[0,\infty)$

and

$$\mathcal{L}(h)(s) = \mathcal{L}(f-g)(s) = \mathcal{L}(f)(s) - \mathcal{L}(g)(s)$$

i.e. $\mathcal{L}(h)(s) = 0$

We have $\displaystyle\int_0^\infty h(t)e^{-st}dt = 0$.

This is true only for $h(t) = 0$

we'll accept as it is.

$\longrightarrow$ $f(t) = g(t)$. $\square$

$\mathcal{L}$-Transform Pairs

$$F(s) = \mathcal{L}(f)(s) = \mathcal{L}\{f(t)\}(s), \quad s > a$$

$\mathcal{L}$–transform pairs:

- $f(t)$ determines $F(s)$ uniquely in $s > a$
- $F(s)$ determines $f(t)$ uniquely in $0 \leq t < \infty$ except at discontinuity points.

Definition of the Inverse Laplace Transform

Def.: Given $F(s)$ and $f(t)$ s.t. $F(s) = \mathcal{L}(f)(s)$, then $f(t)$ is called the inverse Laplace ($\mathcal{L}$) transform of $F(s)$, and is denoted by

$$f(t) = \mathcal{L}^{-1}(F)(t) = \mathcal{L}^{-1}\{F(s)\}(t)$$

$$F = \mathcal{L}(f) \quad \Leftrightarrow \quad f = \mathcal{L}^{-1}(F)$$

Table of Inverse $\mathcal{L}$-Transform

$$F = \mathcal{L}(f) \quad \Leftrightarrow \quad f = \mathcal{L}^{-1}(F)$$

$F(s)$	$\mathcal{L}^{-1}\{F(s)\}(t)$
$\frac{1}{s-c}$	e^{ct}
$\frac{1}{(s-c)^k}$	$\frac{t^{k-1}}{(k-1)!}e^{ct}$
$\frac{1}{(s-\alpha)^2+\beta^2}$	$\frac{e^{\alpha t}\sin\beta t}{\beta}$
$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$	$e^{\alpha t}\cos\beta t$

Exercise 5.3.2

2. Compute the inverse Laplace transform of $Y(s) = \frac{1}{3-5s}$.

2. Adjust as follows.

$$Y(s) = \frac{2}{3-5s} = -\frac{2}{5} \cdot \frac{1}{s-3/5}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{-\frac{2}{5} \cdot \frac{1}{s-3/5}\right\} \\ &= -\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3/5}\right\} \\ &= -\frac{2}{5}e^{(3/5)t} \end{aligned}$$

Exercise 5.3.4

4. Compute the inverse Laplace transform of $Y(s) = \frac{5s}{s^2+9}$

4. Adjust as follows.

$$Y(s) = \frac{5s}{s^2+9} = 5 \cdot \frac{s}{s^2+9}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{5 \cdot \frac{s}{s^2+9}\right\} \\ &= 5\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} \\ &= 5\cos 3t \end{aligned}$$

Exercise 5.3.6

6. Compute the inverse Laplace transform of $Y(s) = \frac{2}{3s^4}$

6. Adjust as follows.

$$Y(s) = \frac{2}{3s^4} = \frac{1}{9} \cdot \frac{3!}{s^4}$$

Thus, by linearity,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{9} \cdot \frac{3!}{s^4}\right\} \\ &= \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} \\ &= \frac{1}{9}t^3 \end{aligned}$$

Exercise 5.3.7

7. Compute the inverse Laplace transform of $Y(s) = \frac{3s+2}{s^2+25}$

7. Adjust as follows:

$$\begin{aligned} Y(s) &= \frac{3s+2}{s^2+25} \\ &= \frac{3s}{s^2+25} + \frac{2}{s^2+25} \\ &= 3 \cdot \frac{s}{s^2+25} + \frac{2}{5} \cdot \frac{5}{s^2+25}. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{3 \cdot \frac{s}{s^2+25} + \frac{2}{5} \cdot \frac{5}{s^2+25}\right\} \\ &= 3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\} + \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\} \\ &= 3\cos 5t + \frac{2}{5}\sin 5t. \end{aligned}$$

Exercise 5.3.9

9. Compute the inverse Laplace transform of $Y(s) = \frac{1}{3-4s} + \frac{3-2s}{s^2+49}$

9. Adjust as follows:

$$\begin{aligned} Y(s) &= \frac{1}{3-4s} + \frac{3-2s}{s^2+49} \\ &= \frac{1}{-4} \cdot \frac{1}{s-3/4} + \frac{3}{s^2+49} - \frac{2s}{s^2+49} \\ &= -\frac{1}{4} \cdot \frac{1}{s-3/4} + \frac{3}{7} \cdot \frac{7}{s^2+49} \\ &\quad - 2 \cdot \frac{s}{s^2+49}. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{ -\frac{1}{4} \cdot \frac{1}{s-3/4} + \frac{3}{7} \cdot \frac{7}{s^2+49} \right. \\ &\quad \left. - 2 \cdot \frac{s}{s^2+49} \right\} \\ &= -\frac{1}{4} \mathcal{L}^{-1}\left\{ \frac{1}{s-3/4} \right\} + \frac{3}{7} \mathcal{L}^{-1}\left\{ \frac{7}{s^2+49} \right\} \\ &\quad - 2\,\mathcal{L}^{-1}\left\{ \frac{s}{s^2+49} \right\} \\ &= -\frac{1}{4} e^{(3/4)t} + \frac{3}{7} \sin 7t - 2\cos 7t. \end{aligned}$$

Exercise 5.3.11

11. Compute the inverse Laplace transform of $Y(s) = \frac{5}{(s+2)^3}$

11. Note the transform pair:

$$t^2 \Longleftrightarrow \frac{2}{s^3}$$

By Proposition 2.12,

$$e^{-2t}t^2 \Longleftrightarrow \frac{2}{(s+2)^3}.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{5}{(s+2)^3}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{5}{2}\cdot\frac{2}{(s+2)^3}\right\} \\ &= \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^3}\right\} \\ &= \frac{5}{2}e^{-2t}t^2. \end{aligned}$$

Exercise 5.3.14

14. Compute the inverse Laplace transform of $Y(s) = \frac{4(s-1)}{(s-1)^2+4}$

14. Note the transform pair.

$$\cos 2t \leftrightarrow \frac{s}{s^2+4}$$

By Proposition 2.12,

$$e^t \cos 2t \leftrightarrow \frac{s-1}{(s-1)^2+4}.$$

Hence,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{4(s-1)}{(s-1)^2+4}\right\} \\ &= 4\,\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} \\ &= 4e^t \cos 2t. \end{aligned}$$

Exercise 5.3.15

15. Compute the inverse Laplace transform of $Y(s) = \frac{2s-3}{(s-1)^2+5}$

15. Note the transform pairs:

$$\cos\sqrt{5}t \Longleftrightarrow \frac{s}{s^2+5}$$
$$\sin\sqrt{5}t \Longleftrightarrow \frac{\sqrt{5}}{s^2+5}$$

By Proposition 2.12,

$$e^t\cos\sqrt{5}t \Longleftrightarrow \frac{s-1}{(s-1)^2+5}$$
$$e^t\sin\sqrt{5}t \Longleftrightarrow \frac{\sqrt{5}}{(s-1)^2+5}.$$

Thus,

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{2s-3}{(s-1)^2+5}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2s-2}{(s-1)^2+5} - \frac{1}{(s-1)^2+5}\right\} \\
&= \mathcal{L}^{-1}\left\{2\cdot\frac{s-1}{(s-1)^2+5} \right. \\
&\qquad \left. - \frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{(s-1)^2+5}\right\} \\
&= 2\,\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+5}\right\} \\
&\qquad - \frac{1}{\sqrt{5}}\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{(s-1)^2+5}\right\} \\
&= 2e^t\cos\sqrt{5}t - \frac{1}{\sqrt{5}}e^t\sin\sqrt{5}t \\
&= e^t(2\cos\sqrt{5}t - \frac{\sqrt{5}}{5}\sin\sqrt{5}t).
\end{aligned}$$

Exercise 5.3.17

17. Compute the inverse Laplace transform of $Y(s) = \frac{3s+2}{s^2+4s+29}$

17. Complete the square.

$$Y(s) = \frac{3s+2}{s^2+4s+29} = \frac{3s+2}{(s+2)^2+25}$$

Note the transform pairs.

$$\cos 5t \Longleftrightarrow \frac{s}{s^2+25}$$
$$\sin 5t \Longleftrightarrow \frac{5}{s^2+25}$$

By Proposition 2.12,

$$e^{-2t}\cos 5t \Longleftrightarrow \frac{s+2}{(s+2)^2+25}$$
$$e^{-2t}\sin 5t \Longleftrightarrow \frac{5}{(s+2)^2+25}.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{3s+2}{(s+2)^2+25}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{3s+6}{(s+2)^2+25} - \frac{4}{(s+2)^2+25}\right\} \\ &= \mathcal{L}^{-1}\left\{3\cdot\frac{s+2}{(s+2)^2+25} \right. \\ &\qquad \left. - \frac{4}{5}\cdot\frac{5}{(s+2)^2+25}\right\} \\ &= 3\,\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+25}\right\} \\ &\quad - \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^2+25}\right\} \\ &= 3e^{-2t}\cos 5t - \frac{4}{5}e^{-2t}\sin 5t \\ &= e^{-2t}(3\cos 5t - \frac{4}{5}\sin 5t). \end{aligned}$$

Inverse $\mathcal{L}$-Transform of Rational Functions

Inverse $\mathcal{L}$–Transform of Rational Functions

Form: $F(s) = \frac{P(s)}{Q(s)}$

- $P(s)$, $Q(s)$: polynomials
- degree of $P <$ degree of Q

Assume $Q(s)$ has k **distinct roots**

Partial Fraction Decomposition (PFD):

$$F(s) = \sum_{\{\lambda\}} F_\lambda(s)$$

$F_\lambda(s)$: contribution from root λ

Linearity $\Rightarrow$

$$\mathcal{L}^{-1}(F)(t) = \sum_{\{\lambda\}} \mathcal{L}^{-1}(F_\lambda)(t)$$

Let m be the multiplicity of λ. Set $Q_\lambda(s) = Q(s)/(s-\lambda)^m \Rightarrow Q_\lambda(\lambda) \neq 0$

Simple Root: $(m = 1)$

Simple Root: $(m = 1)$

$$F_\lambda(s) = \frac{A}{s-\lambda}, \quad A = \frac{P(\lambda)}{Q_\lambda(\lambda)}$$
$$\Rightarrow \mathcal{L}^{-1}(F_\lambda)(t) = Ae^{\lambda t}$$

Complex Case: Assume $\lambda = \alpha + i\beta$, $\overline{\lambda} = \alpha - i\beta$ are a complex conjugate pair of simple roots

$$\Rightarrow F_\lambda(s) + F_{\overline{\lambda}}(s) = \frac{A}{s-\lambda} + \frac{\overline{A}}{s-\overline{\lambda}}$$

$$\begin{aligned} \Rightarrow \mathcal{L}^{-1}(F_\lambda + F_{\overline{\lambda}})(t) &= Ae^{\lambda t} + \overline{A}e^{\overline{\lambda}t} \\ &= 2\text{Re}(Ae^{\lambda t}) \end{aligned}$$

Real version: let $A = a + ib$

$$\Rightarrow F_\lambda(s) + F_{\overline{\lambda}}(s) = \frac{2a(s-\alpha) - 2b\beta}{(s-\alpha)^2 + \beta^2}$$

$$\Rightarrow \mathcal{L}^{-1}(F_\lambda + F_{\overline{\lambda}})(t) =$$

$$2e^{\alpha t}(a\cos\beta t - b\sin\beta t)$$

Multiple Root: $(m > 1)$

Multiple Root: $(m > 1)$

$$F_\lambda(s) = \frac{A_m}{s-\lambda} + \frac{A_{m-1}}{(s-\lambda)^2} + \cdots + \frac{A_2}{(s-\lambda)^{m-1}} + \frac{A_1}{(s-\lambda)^m}$$

$$\Rightarrow \mathcal{L}^{-1}(F_\lambda)(s) = e^{\lambda t}[A_m + A_{m-1}t + \cdots + A_1 t^{m-1}/(m-1)!]$$

For multiple complex pairs $\lambda, \overline{\lambda}$:

$$\mathcal{L}^{-1}(F_\lambda + F_{\overline{\lambda}})(t) = 2\Big[\text{Re}(A_m e^{\lambda t}) + t\text{Re}(A_{m-1}e^{\lambda t}) + \cdots + \frac{t^{m-2}\text{Re}(A_2 t e^{\lambda t})}{(m-2)!} + \frac{t^{m-1}\text{Re}(A_1 t e^{\lambda t})}{(m-1)!}\Big]$$

Coefficients:

$$A_j = \frac{1}{(j-1)!}\left[\frac{d^{j-1}}{ds^{j-1}}\left(\frac{P(s)}{Q_\lambda(s)}\right)\right]_{s=\lambda}$$

For $\mathbf{m = 2}$**:**

$$A_1 = \frac{P(\lambda)}{Q_\lambda(\lambda)}, \quad A_2 = \left[\frac{d}{ds}\left(\frac{P(s)}{Q_\lambda(s)}\right)\right]_{s=\lambda}$$

Example 1

Ex. 1: $F(s) = \frac{s+9}{s^2-2s-3} = \frac{s+9}{(s+1)(s-3)}$

Roots: $\lambda_1 = -1,\ \lambda_2 = 3 \rightarrow$

$$F(s) = F_{-1}(s) + F_3(s)$$

$$F_{-1}(s) = \frac{A}{s+1},\quad F_3(s) = \frac{B}{s-3}$$

$$Q_{-1}(s) = \frac{(s+1)(s-3)}{s+1} = s-3$$

$$\Rightarrow A = \frac{s+9}{s-3}\Big|_{s=-1} = -2$$

$$Q_3(s) = \frac{(s+1)(s-3)}{s-3} = s+1$$

$$\Rightarrow B = \frac{s+9}{s+1}\Big|_{s=3} = 3$$

$$\Rightarrow F(s) = \frac{-2}{s+1} + \frac{3}{s-3}$$

$$\Rightarrow \mathcal{L}^{-1}(t) = -2e^{-t} + 3e^{3t}$$

Example 1(cont.)

Other methods for finding A, B:
(see text, Sec. 5.3, Example 3.6)

$$\frac{s+9}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3}$$

$$\Rightarrow\ s+9 = A(s-3) + B(s+1) \quad (2)$$

Substitution method:
Substitute two values for s in (2):

$$s = 3 \ \Rightarrow\ 12 = 4B \ \Rightarrow\ B = 3$$
$$s = -1 \ \Rightarrow\ 8 = -4A \ \Rightarrow\ A = -2$$

Coefficient method: Rewrite (2) as

$$s + 9 = (A+B)s + (-3A+B)$$

Equate coefficients of powers of s:

$$\Rightarrow \left\{ \begin{array}{ccc} 1 & = & A+B \\ 9 & = & -3A+B \end{array} \right\}$$

$$\Rightarrow \left\{ \begin{array}{ccc} A & = & -2 \\ B & = & 3 \end{array} \right\}$$

Example 2

Ex. 2:

$$Y(s) = \frac{s-2}{s^2-2s-3} = \frac{s-2}{(s+1)(s-3)}$$

$$= \frac{A}{s+1} + \frac{B}{s-3}$$

$$A = \left.\frac{s-2}{s-3}\right|_{s=-1} = \frac{3}{4}$$

$$B = \left.\frac{s-2}{s+1}\right|_{s=3} = \frac{1}{4}$$

$$\Rightarrow Y(s) = \frac{1}{4}\left(\frac{3}{s+1} + \frac{1}{s-3}\right)$$

$$\Rightarrow \mathcal{L}^{-1}(Y)(t) = \frac{1}{4}(3e^{-t} + e^{3t})$$

Example 3

Ex. 3: $F(s) = \frac{1}{s^2+4s+13} = \frac{1}{(s+2)^2+9}$

This is of the form

$$\frac{1}{(s-\alpha)^2+\beta^2} \quad (\alpha = -2,\ \beta = 3)$$

with inverse transform (see table)

$$(1/\beta)e^{\alpha t}\sin \beta t$$

$$\Rightarrow \mathcal{L}^{-1}(F)(t) = (1/3)e^{-2t}\sin 3t$$

See text, Sec. 5.3, Example 3.6, for coefficient and substitution methods.

Example 4

Ex. 4: $F(s) = \frac{2s^2+s+13}{(s-1)[(s+1)^2+4]}$
(see text, Sec. 5.3, Example 3.9)
$(s+1)^2+4 = (s+1+2i)(s+1-2i)$
$\Rightarrow$ roots of $Q(s)$:

$$\lambda_1 = 1,\ \lambda_2 = -1+2i,\ \lambda_3 = \overline{\lambda_2}$$

$$F_{\lambda_1}(s) = \frac{A}{s-1},\ A = \frac{2s^2+s+13}{(s+1)^2+4}\Big|_{s=1} = 2$$

$$\Rightarrow \mathcal{L}^{-1}(F_{\lambda_1})(t) = 2e^t$$

Work on λ_2: $F_{\lambda_2}(s) = \frac{B}{s+1-2i}$

$$B = \frac{2s^2+s+13}{(s-1)(s+1+2i)}\Big|_{s=-1+2i} = \frac{2(1-4i-4)+(-1+2i)+13}{(-2+2i)4i} = \frac{6-6i}{-8-8i} = -\frac{3}{4}\frac{1-i}{1+i} = \frac{3i}{4}$$

$$\Rightarrow F_{\lambda_2}(s) + F_{\overline{\lambda_2}}(s) = \tfrac{3}{4}\left(\tfrac{i}{s+1-2i} - \tfrac{i}{s+1+2i}\right) = \tfrac{-3}{(s+1)^2+4}$$

$$\Rightarrow \mathcal{L}^{-1}(F_{\lambda_2} + F_{\overline{\lambda_2}})(t) = -\tfrac{3}{2}e^{-t}\sin 2t$$

$$\Rightarrow \mathcal{L}^{-1}(F)(t) = 2e^t - (3/2)e^{-t}\sin 2t$$

Example 5

Ex. 5: $Y(s) = \frac{s^2+s+4}{(s^2+1)(s^2+4)}$

$$\left.\begin{aligned} s^2+1 &= (s-i)(s+i) \\ s^2+4 &= (s-2i)(s+2i) \end{aligned}\right\} \Rightarrow \text{roots:}$$

$$\lambda_1 = i,\ \lambda_2 = -i\ \lambda_3 = 2i,\ \lambda_4 = -2i$$

$$Y_{\lambda_1}(s) = \frac{A}{s-i}$$

$$\begin{aligned} A &= \left.\frac{s^2+s+4}{(s+i)(s^2+4)}\right|_{s=i} \\ &= \frac{3+i}{6i} = \frac{1}{6}(1-3i) \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathcal{L}^{-1}(Y_{\lambda_1} + Y_{\overline{\lambda}_1})(t) &= 2\text{Re}(Ae^{it}) \\ &= \frac{1}{3}(\cos t + 3\sin t) \end{aligned}$$

$$Y_{\lambda_3}(s) = \frac{B}{s-2i}$$

$$\begin{aligned} A &= \left.\frac{s^2+s+4}{(s^2+1)(s+2i)}\right|_{s=2i} \\ &= \frac{2i}{(-3)4i} = -\frac{1}{6} \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathcal{L}^{-1}(Y_{\lambda_3} + Y_{\overline{\lambda}_3})(t) &= 2\text{Re}(Be^{2it}) \\ &= -\frac{1}{3}\cos 2t \end{aligned}$$

$$\mathcal{L}^{-1}(Y)(t) = (1/3)(\cos t + 3\sin t - \cos 2t)$$

Example 6

Ex. 6: $Y(s) = \frac{1}{(s+1)(s-1)^2}$

Roots: $\lambda_1 = -1,\ \lambda_2 = 1\ (m = 2)$

$Y_{-1}(s) = \dfrac{A}{s+1},\ A = \dfrac{1}{(s-1)^2}\Big|_{s=-1} = \dfrac{1}{4}$

$$Y_1(s) = \frac{B_1}{(s-1)^2} + \frac{B_2}{s-1}$$

$$\begin{aligned} B_1 &= \frac{1}{s+1}\Big|_{s=1} = \frac{1}{2} \\ B_2 &= \left(\frac{d}{ds}\frac{1}{s+1}\right)\Big|_{s=1} = -\frac{1}{4} \end{aligned}$$

$$Y(s) = \frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}$$

$$\mathcal{L}^{-1}(Y)(t) = \frac{1}{4}(e^{-t} - e^{t} + 2te^{t})$$

Example 7

Ex. 7: $Y(s) = \frac{s}{(s^2+2s+2)(s^2+4)}$

$Q(s) = [(s+1)^2+1](s^2+4)$: factorize $(s+1)^2+1 = (s+1-i)(s+1+i)$, $s^2+4 = (s-2i)(s+2i) \Rightarrow$ roots $\lambda_1 = -1+i$, $\lambda_2 = \overline{\lambda}_1$, $\lambda_3 = 2i$, $\lambda_4 = \overline{\lambda}_3$

$$Y_{\lambda_1}(s) = \frac{A}{s+1-i}, \quad A = \frac{s}{(s+1+i)(s^2+4)}\Big|_{s=-1+i} = \frac{-1+i}{2i((1-i)^2+4)}$$

$$= \frac{-1+i}{2i(4-2i)} = \frac{1}{4}\frac{-1+i}{1+2i} = \frac{1}{4}\frac{1}{5}(-1+i)(1-2i) = \frac{1}{20}(1+3i)$$

$$\Rightarrow \mathcal{L}^{-1}(Y_{\lambda_1} + Y_{\overline{\lambda}_1})(t) = 2e^{-t}\text{Re}(\frac{1}{20}(1+3i)e^{it}) = \frac{1}{10}e^{-t}(\cos t - 3\sin t)$$

$$Y_{\lambda_3}(s) = \frac{B}{s-2i}, \quad B = \frac{s}{(s^2+2s+2)(s+2i)}\Big|_{s=2i} = \frac{2i}{(-2+4i)4i}$$

$$= -\frac{1}{4}\frac{1}{1-2i} = -\frac{1}{20}(1+2i)$$

$$\Rightarrow \mathcal{L}^{-1}(Y_{\lambda_3} + Y_{\overline{\lambda}_3})(t) = 2\text{Re}(-\frac{1}{20}(1+2i)e^{2it}) = -\frac{1}{10}(\cos 2t - 2\sin 2t)$$

$$\Rightarrow \mathcal{L}^{-1}(Y)(t) = \frac{1}{10}(e^{-t}\cos t - 3e^{-t}\sin t - \cos 2t + 2\sin 2t)$$