Math 3331 Differential Equations

5.4 Using the $\mathcal{L}$-Transform to Solve ODE

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- Basic Idea
- Examples

Basic Idea

Basic Idea: $\left\{\begin{array}{c}\text{IVP}\\ \text{for } y(t):\\ \text{ODE+IC}\end{array}\right\} \xrightarrow{\mathcal{L}} \left\{\begin{array}{c}\text{algebraic}\\ \text{equation}\\ \text{for } Y(s)\end{array}\right\} \xrightarrow{\text{solve}} Y(s) \xrightarrow{\mathcal{L}^{-1}} y(t)$

Laplace

find Y(s)

inverse

Example 8

Ex. 8: $y'' + y = \cos 2t$
$y(0) = 0,\ y'(0) = 1$

$\mathcal{L}$–transform ODE:

$$\mathcal{L}(y'' + y) = \mathcal{L}\{\cos 2t\}$$

$$\begin{aligned} \mathcal{L}(y'') &= s^2Y - sy(0) - y'(0) \\ &= s^2Y - 1 \\ \mathcal{L}(y) &= Y \end{aligned}$$

$$\Rightarrow \mathcal{L}(y'' + y) = (s^2 + 1)Y - 1$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2 + 4}$$

$$\Rightarrow (s^2 + 1)Y - 1 = \frac{s}{s^2 + 4}$$

$$\Rightarrow Y(s) = \frac{1}{s^2 + 1} + \frac{s}{(s^2 + 1)(s^2 + 4)}$$

$$= \frac{s^2 + s + 4}{(s^2 + 1)(s^2 + 4)}$$

$y(t) = \mathcal{L}^{-1}(Y)(t)$. From Ex. 5 $\Rightarrow$

$$y(t) = \frac{1}{3}(\cos t + 3\sin t - \cos 2t)$$

Example 9

Ex. 9: $y'' - 2y' - 3y = 0$

$$y(0) = 1,\ y'(0) = 0$$

$$\begin{aligned} \mathcal{L}(y'') &= s^2Y - s \\ \mathcal{L}(y') &= sY - 1 \end{aligned}$$

$$\Rightarrow \mathcal{L}(y''-2y'-3y) = (s^2-2s-3)Y-s+2 = 0$$

$$\Rightarrow\ Y(s) = \frac{s-2}{s^2-2s-3},\ y(t) = \mathcal{L}^{-1}(Y)(t)$$

From Ex. 2: $y(t) = (1/4)(e^{3t} + 3e^{-t})$

Without Laplace:

$$y'' - 2y' - 3y = 0 \qquad y(0) = 1$$
$$y'(0) = 0$$

$\hookrightarrow$ $\lambda^2 - 2\lambda - 3 = 0$

$(\lambda+1)(\lambda-3) = 0 \Rightarrow \lambda = -1, 3$

$\Rightarrow \quad y(t) = c_1 e^{3t} + c_2 e^{-t}$

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ y'(0) &= 3c_1 - c_2 = 0 \end{aligned}\right\} \rightarrow \begin{aligned} c_1 &= \frac{1}{4} \\ c_2 &= \frac{3}{4} \end{aligned}$$

$$\boxed{y(t) = \frac{1}{4} e^{3t} + \frac{3}{4} e^{-t}}$$

match

Example 10

Ex. 10: $y'' - y = e^t$, $y(0) = y'(0) = 0$

$\mathcal{L}(y'' - y) = (s^2 - 1)Y$, $\mathcal{L}\{e^t\} = \dfrac{1}{s-1}$

$\Rightarrow (s^2 - 1)Y = 1/(s-1)$

$\Rightarrow Y(s) = \frac{1}{(s^2-1)(s-1)} = \frac{1}{(s+1)(s-1)^2}$

Ex. 6 $\Rightarrow y(t) = (e^{-t} - e^t + 2te^t)/4$

$$Y(s) = \frac{1}{(s-1)^2(s+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1}$$

$$\Rightarrow \quad A(s^2-1) + B(s+1) + C(s-1)^2 = 1$$

$$(A+C)s^2 + (B-2C)s + (-A+B+C) = 1$$

$$+\begin{cases} A+C=0 \\ B-2C=0 \\ -A+B+C=1 \end{cases}$$

$$2B = 1 \longrightarrow B = \frac{1}{2}$$

$$\frac{1}{2} - 2C = 0$$

$$C = \frac{1}{4} \Rightarrow A = -\frac{1}{4}$$

$$Y(s) = \frac{-1}{4}\cdot\frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{(s-1)^2} + \frac{1}{4}\cdot\frac{1}{s+1}$$

$$\Rightarrow \boxed{y(t) = -\frac{1}{4}e^{t} + \frac{1}{2}te^{t} + \frac{1}{4}e^{-t}}$$

Without Laplace:

$$y'' - y = e^t \quad , \quad y(0) = y'(0) = 0$$

- $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow \boxed{y_H = C_1 e^t + C_2 e^{-t}}$

- $y_p = A t e^t = \frac{1}{2} t e^t$
  $y_p' = A e^t + A t e^t$
  $y_p'' = A e^t + A e^t + A t e^t$
  $\Rightarrow y_p'' - y_p = e^t$
  $2A e^t = 1 e^t$
  $\underbrace{2A}_{} = 1 \Rightarrow \boxed{A = \frac{1}{2}}$

- $y(t) = C_1 e^t + C_2 e^{-t} + \frac{1}{2} t e^t \Rightarrow C_1 + C_2 = 0$
  $y'(t) = C_1 e^t - C_2 e^{-t} + \frac{1}{2} e^t + \frac{1}{2} t e^t \Rightarrow C_1 - C_2 + \frac{1}{2} = 0$

$$\Rightarrow \quad 2c_1 = -\frac{1}{2} \quad \Rightarrow \quad c_1 = -\frac{1}{4}, \quad c_2 = \frac{1}{4}$$

$$\Rightarrow \boxed{y(t) = -\frac{1}{4}e^{t} + \frac{1}{4}e^{-t} + \frac{1}{2}te^{t}}$$

<u>Match</u>.

Example 11

Ex. 11: $y'' + 2y' + 2y = \cos 2t,\ y(0) = 0,\ y'(0) = 1$

$$\mathcal{L}(y'' + 2y' + 2y) = (s^2 Y - 1) + 2(sY) + 2Y = (s^2 + 2s + 2)Y - 1$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4} \Rightarrow (s^2 + 2s + 2)Y - 1 = \frac{s}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{1}{s^2+2s+2} + \frac{s}{(s^2+2s+2)(s^2+4)}$$

$$\mathcal{L}^{-1}\{\frac{1}{s^2+2s+2}\} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2+1}\} = e^{-t}\sin t$$

From Ex. 7: $\mathcal{L}^{-1}\{\frac{s}{(s^2+2s+2)(s^2+4)}\}$

$$= \frac{1}{10}(e^{-t}\cos t - 3e^{-t}\sin t - \cos 2t + 2\sin 2t)$$

$$\Rightarrow y(t) = \frac{1}{10}(e^{-t}\cos t + 7e^{-t}\sin t - \cos 2t + 2\sin 2t)$$