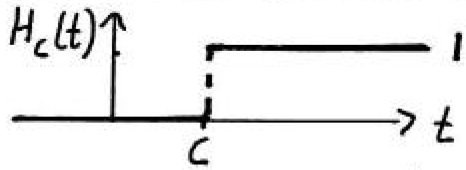
We have been using this actually: $f(t) \underset{\text{Same}}{\longleftrightarrow} f(t)H(t)$, $t \geq 0$

5.5 Discontinuous Forcing Functions

Heaviside step function ($c \geq 0$)

$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

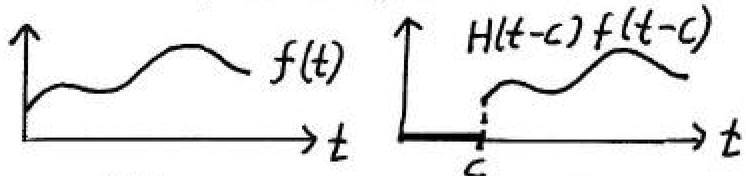


shifted $\rightarrow$ $H_c(t) \equiv H(t-c) = \begin{cases} 1 & \text{for } t \geq c \\ 0 & \text{for } t < c \end{cases}$

Since we restrict to $t \geq 0$, identify

$H_0(t) \equiv H(t) = 1 \quad (\text{in } t \geq 0)$

t-shift property of $\mathcal{L}$:



Laplace and its Inverse

$\mathcal{L}\{H(t-c)f(t-c)\} = e^{-cs}F(s)$

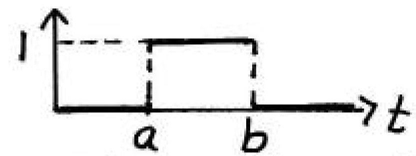
or $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = H(t-c)f(t-c)$

$\Rightarrow \mathcal{L}\{H_c(t)\} = e^{-cs}\mathcal{L}(1) = \dfrac{e^{-cs}}{s}$

Boxcar (Interval) function

$H_{ab}(t) = H_a(t) - H_b(t) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } a \leq t < b \\ 0 & \text{for } t \geq b \end{cases}$

$(a < b)$



$\mathcal{L}\{H_{ab}(t)\}(s) = (e^{-as} - e^{-bs})/s$

Piecewise Defined Functions

Ex. 1: $g(t) = \begin{cases} 2t & \text{for } 0 \leq t < 1 \\ 2 & \text{for } 1 \leq t < \infty \end{cases}$

Use boxcars:

$g(t) = 2t[1 - H(t-1)] + 2H(t-1)$

$= 2t - 2H(t-1)(t-1)$

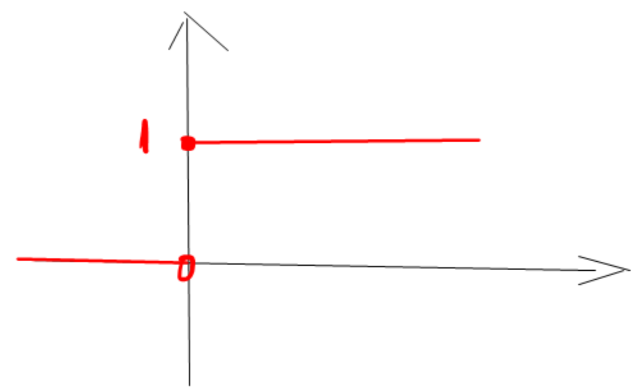
$\Rightarrow \mathcal{L}(g)(s) = 2\mathcal{L}\{t\}(s) - 2\mathcal{L}\{H(t-1)(t-1)\}$

$= 2/s^2 - 2e^{-s}\mathcal{L}\{t\}(s)$

$= 2/s^2 - 2e^{-s}/s^2$

Heaviside Step Function

- $H(t) = H_0(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$



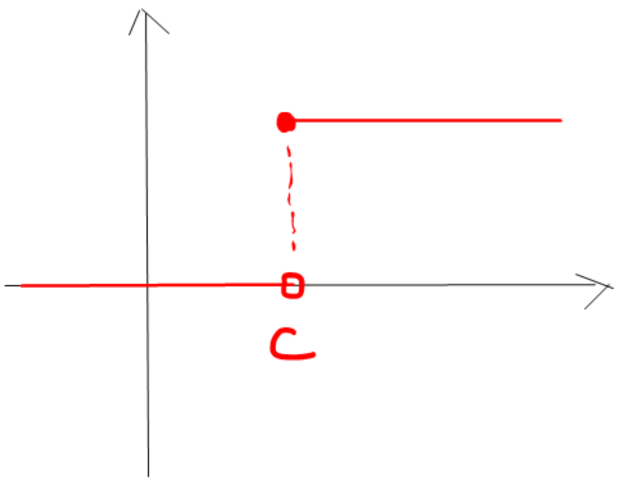
$$\boxed{\mathcal{L}(H(t))(s) = \mathcal{L}(1)(s) = \frac{1}{s}}$$

Note that any $f(t)$ on $[0, \infty)$ can be seen as $f(t)H(t)$

$$\boxed{\mathcal{L}(f(t))(s) = \mathcal{L}\left(f(t)H(t)\right)(s) = F(s)}$$

- $H_c(t) = H(t-c) = \begin{cases} 1, & t \geq c \\ 0, & t < c \end{cases}$

(shift c units horizontally)



$$\boxed{\mathcal{L}\big(H(t-c)\big)(s) = e^{-cs} \cdot \frac{1}{s}}$$

Proof: $\mathcal{L}\big(H(t-c)\big)(s) = \int_0^\infty H(t-c)\, e^{-st}\, dt = \int_c^\infty 1 \cdot e^{-st}\, dt$

change of variables $u = t-c$, $du = dt$

$t = u+c$

$$= \int_0^\infty 1 \cdot e^{-s(u+c)}\, du = e^{-sc} \int_0^\infty 1 \cdot e^{-su}\, du = e^{-sc} \cdot \frac{1}{s}$$

$$\boxed{\mathcal{L}\left(f(t-c)\,H(t-c)\right)(s) = e^{-cs}\cdot F(s)}$$

Proof:

$$\mathcal{L}\left(\underline{\underline{f(t-c)\,H(t-c)}}\right)(s) = \int_0^\infty f(t-c)\,\underbrace{H(t-c)}_{=1\ ,\ t\geq c}\,e^{-st}\,dt$$

$$= \int_c^\infty f(t-c)\,e^{-st}\,dt = \int_0^\infty f(u)\,e^{-s(u+c)}\,du$$

$u = t-c$

$$= e^{-cs}\cdot \underbrace{\int_0^\infty f(u)\,e^{-su}\,du}_{F(s)} = \underline{\underline{e^{-cs}\cdot F(s)}}$$

ex.

$$g(t) = \begin{cases} 2t, & t \geq 3 \\ 0, & t < 3 \end{cases}$$

$$\Longrightarrow \quad g(t) = 2t\, H_3(t) = \underset{\text{should match to}}{2\underline{t}}\; H(\overbrace{t-3})$$

$$= 2(\underbrace{t-3} + 3)\, H(t-3)$$

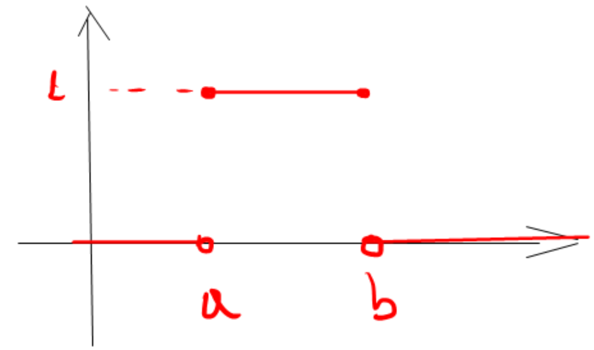
$$= 2(t-3)\, H(\underbrace{t-3}) + 6\, H(\underbrace{t-3}) \cdot 1$$

$\mathcal{L}(t) = \frac{1}{s^2}$

Laplace

$$\boxed{G(s) = 2 \cdot e^{-3s} \cdot \frac{1}{s^2} + 6 \cdot e^{-3s} \cdot \frac{1}{s}}$$

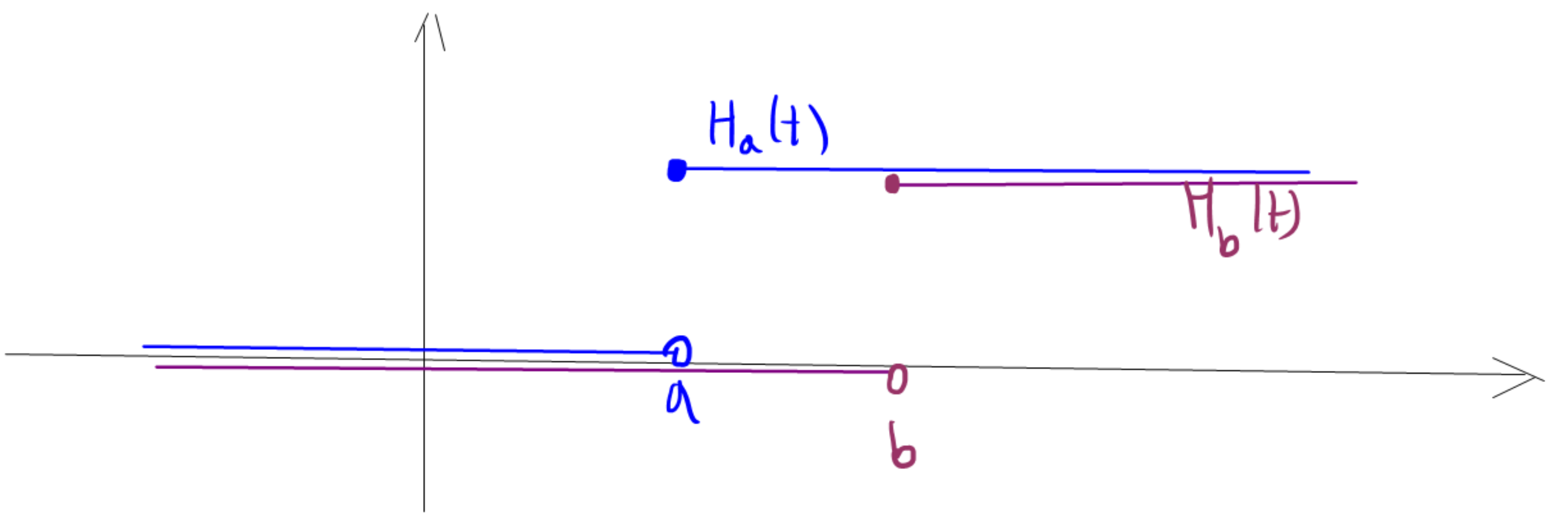
- $$H_{ab}(t) = \begin{cases} 0, & t < a \\ 1, & a \le t \le b \\ 0, & t > b \end{cases}$$



Note that

$$H_{ab}(t) = H_a(t) - H_b(t)$$

$$H_{ab}(t) = H(t-a) - H(t-b)$$



ex

$$f(t) = \begin{cases} 3, & 0 \le t < 4 \\ -5, & 4 \le t < 6 \\ e^{7-t}, & t \geq 6 \end{cases}$$

$$\Rightarrow \quad f(t) = 3 \cdot H_{04} - 5 H_{46} + e^{7-t} H_6$$

$$= 3\big(H(t) - H(t-4)\big) - 5\big(H(t-4) - H(t-6)\big) + e^{7-t} H(t-6)$$

??

rewrite

$$= 3H(t) - 8H(t-4) + 5H(t-6) + e \cdot e^{-(t-6)} \cdot H(t-6)$$

$$\Rightarrow \quad F(s) = 3 \cdot \frac{1}{s} - 8 \cdot e^{-4s} \cdot \frac{1}{s} + 5 \cdot e^{-6s} \cdot \frac{1}{s} + e \cdot \frac{1}{s+1} \cdot e^{-6s}$$

$$\boxed{F(s) = \frac{3}{s} - \frac{8e^{-4s}}{s} + \frac{5e^{-6s}}{s} + \frac{e^{-6s+1}}{s+1}}$$

Inverse $\mathcal{L}$-Transforms of Functions with Exponential Terms

Ex. 2 : $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s^2+1)}\right\}(t)$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}(t-1)$$

$$F(s) \equiv \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\{e^{-s}F(s)\}(t)$$

$$= \mathcal{L}^{-1}\{1/s^2\}(t-1) + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t-1)$$

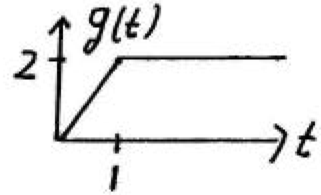
$$= H(t-1)[t-1-\sin(t-1)]$$

IVP's with Discontinuous Forcings

Ex. 3 : $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$

$$g(t) = \begin{cases} 2t & \text{for } 0 \le t < 1 \\ 2 & \text{for } t \ge 1 \end{cases}$$

(as in Ex. 1)



$$\mathcal{L}(y''+y) = (s^2+1)Y - 1$$

$$\mathcal{L}(g) = 2(1-e^{-s})/s^2 \quad \text{(from Ex. 1)}$$

$$\Rightarrow (s^2+1)Y - 1 = 2(1-e^{-s})/s^2$$

$$\Rightarrow Y(s) = \frac{1}{s^2+1} + \frac{2}{s^2(s^2+1)} - \frac{2e^{-s}}{s^2(s^2+1)}$$

$$\Rightarrow y(t) = \sin t + 2(t - \sin t) - 2H(t-1)[t-1-\sin(t-1)] \quad \text{(Ex. 2)}$$

$$\Rightarrow y(t) = -\sin t + 2t[1-H(t-1)] + 2H(t-1)[1+\sin(t-1)]$$

$$= (2t - \sin t)[1-H(t-1)] + [2 + 2\sin(t-1) - \sin t]\,H(t-1)$$

$$= \begin{cases} 2t - \sin t & \text{for } 0 \le t < 1 \\ 2 + 2\sin(t-1) - \sin t & \text{for } t \ge 1 \end{cases}$$

(see text, p. 221, for graph)