Math 3331 Differential Equations

6.2 Runge-Kutta Methods (RKM)

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6.2 Runge-Kutta Methods (RKM)

- 2nd Order RKM
  - 2nd Order RKM: Improved Euler Method
  - 2nd Order RKM: Iteration Scheme
  - Examples Errors in 2nd Order RKM: Second Order
- 4th Order RKM
  - 4th Order RKM: Basic Idea
  - 4th Order RKM: Iteration Scheme
  - Errors in 4th Order RKM: Fourth Order

2nd Order RKM: Improved Euler Method

Failure of Euler Method: Only slope on left end of interval $[t, t+h]$ is used.

Improvement: Given $t, y(t)$,

- compute slope at t

$$s_l = f(t, y(t))$$

$y' = f(t,y)$

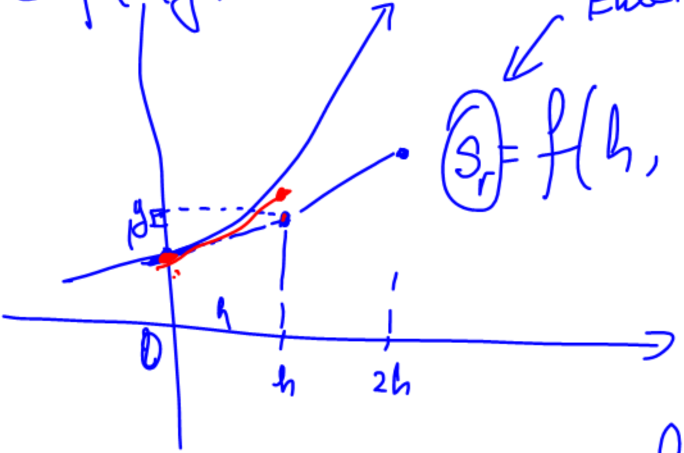
- find slope at $t+h$ via EM

$$\boxed{y_E = y(t) + hs_l}$$
$$s_r = f(t+h, y_E)$$

- approximate $y(t+h)$ via average slope

$$y(t+h) \approx y(t) + h(s_l + s_r)/2$$

$y' = f(t, y)$



Euler Method

$S_r = f(h, y_E)$

$S_l = f(t_0, y_0)$, $S = f(t_0 + h, y_E)$

$y' = y \quad , \quad y(0) = 1 \; , \quad h = 0.5$

Euler Method:

$t_0 = 0$

$y_0 = 1$ slope $= 1$

$t_1 = 0.5$ slope $= 1$

$y_1^{ap} = 1.5$

$t_0 = 0$

$y' = f(t, y)$

$y_0 = 1$

$S_{left} = 1$

$S_R = f(h, y_1^{ap}) = y_1^{ap} = \underline{\underline{1.5}}$

$S_{RKM} = \frac{1+1.5}{2} = \underline{\underline{1.25}}$

$t_0 = 0$

$y_0 = 1$

$S = \underline{1.25}$

$\text{Error} = \left| \sqrt{e} - 1.625 \right|$

$= \left| 0.0237 \right|$

$t_1 = h = 0.5$

$y_1 = y_0 + \text{slope} \cdot h$

$= 1 + 1.25 \cdot (0.5) = \underline{1.625}$

2nd Order RKM: Iteration Scheme

2nd Order RKM

Iteration Scheme

Start: y_0, t_0

For $k = 0$ to $k = N$: Euler Method

$$\begin{aligned} t_{k+1} &= t_k + h \\ s_l &= f(t_k, y_k) \\ s_r &= f(t_{k+1}, y_k + hs_l) \\ y_{k+1} &= y_k + h(s_l + s_r)/2 \end{aligned}$$

Example

$f(t,y) = \text{slope}$

Ex. Approximate the solution to

$$\boxed{y' = t - y,}\ y(0) = 0.5$$

in $0 \le t \le 1$ using $h = 0.25$.

$\text{slope} = t_0 - y_0$

Start: $y_0 = 0.5$, $t_0 = 0$.

Euler

1st step

$t_1 = 0.25$

$$
\begin{aligned}
t_1 &= 0.25 \\
s_l &= t_0 - y_0 = -0.5 \\
s_r &= t_1 - (y_0 + h s_l) = -0.125 \\
y_1 &= y_0 + h(s_l + s_r)/2 = 0.4219 \\
t_2 &= 0.5 \\
s_l &= t_1 - y_1 = -0.1719 \\
s_r &= t_2 - (y_1 + h s_l) = 0.1211 \\
y_2 &= y_1 + h(s_l + s_r)/2 = 0.4155
\end{aligned}
$$

average

$$
\begin{aligned}
t_3 &= 0.75 \\
s_l &= t_2 - y_2 = 0.0845 \\
s_r &= t_3 - (y_2 + h s_l) = 0.3134 \\
y_3 &= y_2 + h(s_l + s_r)/2 = 0.4653 \\
t_4 &= 1 \\
s_l &= t_3 - y_3 = \cancel{0.0845}\ 0.2847 \\
s_r &= t_4 - (y_3 + h s_l) = \cancel{0.3134}\ 0.4635 \\
y_4 &= y_3 + h(s_l + s_r)/2 = \cancel{0.4653}\ 0.5588
\end{aligned}
$$

Euler $y_1 = y_0 + \text{slope} \cdot h$

Errors in 2nd Order RKM: Second Order

Ex.: $y' = t - y, \quad y(0) = 0.5$
Approximate $y(1)$ for stepsizes

$$h = 1/m, \quad m = 1, 2, 4, 8, 16, 32$$

Exact Value: $y(1) = 0.551819$

Error: $E(h) = |y(1) - y_m|$

h	y_m	$E(h)$
1	0.75	0.198181
1/2	0.585938	0.034118
1/4	0.558794	0.006974
1/8	0.553400	0.001581
1/16	0.552196	0.000377
1/32	0.551911	0.000092

$$E(h/2) \approx E(h)/4 \;\Rightarrow\; E(h) \approx C\,h^2$$

Theorem: There $\exists\, C > 0$ s.t.

$$E(h) \leq C\,h^2$$

(2nd order RKM is second order method)

4th Order RKM: Basic Idea

4th Order RKM

Idea: Given t and $y = y(t)$, compute slopes s_1, s_2, s_3, s_4 at four carefully chosen points s.t. error is minimized.

Approximation:

$$y(t+h) \approx y + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4)$$

4th Order RKM: Iteration Scheme

4th Order RKM

$y' = t - y$

$f(1,4) = 1-4 = -3$

Iteration $\mathbf{k \to k+1}$:

$$
\begin{aligned}
s_1 &= f(t_k, y_k) \\
s_2 &= f(t_k + h/2, y_k + hs_1/2) \\
s_3 &= f(t_k + h/2, y_k + hs_2/2) \\
s_4 &= f(t_k + h, y_k + hs_3) \\
y_{k+1} &= y_k + \frac{h}{6}(s_1 + 2s_2 + 2s_3 + s_4) \\
t_{k+1} &= t_k + h
\end{aligned}
$$

Errors in 4th Order RKM: Fourth Order

For each step:

$E(h_i) \leq C_i \cdot h^5$

$\Rightarrow E(h) = \sum_{i=1}^{n} E(h_i) \leq C \cdot n \cdot h^5$

$E(h) \leq C \cdot h^4$

Ex.: $y' = t - y$, $y(0) = 0.5$, $y(1) \approx y_m$

$m = 1, 2, 4, 8, 16, 32$, $h = 1/m$

Exact Value: $y(1) = 0.551819162$

Error: $E(h) = |y(1) - y_m|$

h	y_m	$E(h)$
1	0.5625	0.010680838
1/2	0.552256266	0.000437105
1/4	0.551841299	0.000022137
1/8	0.551820408	0.000001246
1/16	0.551819236	0.000000074
1/32	0.551819166	0.000000005

$E(h/2) \approx E(h)/16 \;\Rightarrow\; E(h) \approx C\,h^4$

Theorem: There $\exists\, C > 0$ s.t.

$$E(h) \leq C\,h^4$$

(4th order RKM is fourth order method)

$n = \frac{b-a}{h}$