Math 3331 Differential Equations

8.1 Introduction to Systems

Blerina Xhabli

Department of Mathematics, University of Houston

blerina@math.uh.edu
math.uh.edu/∼blerina/teaching.html

8.1 Introduction to Systems Definitions and Examples

- Definitions
  - System of First Order ODEs
  - IVP: Existence and Uniqueness of Solution
- Examples
- Reduction of Higher Order Equations
- Worked out Examples from Exercises:
  - 1, 2, 7

SIR Model of an Epidemics

N - population $= S(t) + I(t) + R(t)$

- $S(t)$ – suspectible group (people prone to get the disease)
- $I(t)$ – infected group (infected people)
- $R(t)$ – recovered group (immuned people)

Rate of Change of these groups:

- $\frac{dS}{dt} = -a\underbrace{SI}_{\text{contact}}, \quad a>0$

- $\frac{dI}{dt} = a\underbrace{SI}_{\text{contact}} - b\underbrace{I}_{\text{recovered}}$

- $\frac{dR}{dt} = \underbrace{bI}_{\text{recovered}}$

Note that $\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0.$

$$\boxed{\begin{aligned} S' &= -aSI \\ I' &= aSI - bI \\ R' &= bI \end{aligned}}$$

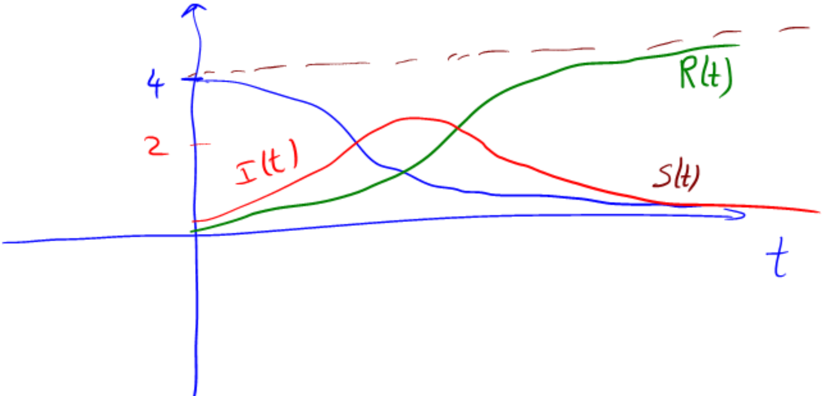
1st order nonlinear system, autonomous.

Given $a = b = 1$, consider

$$S' = -SI \qquad S(0) = 4$$

$$I' = +SI - I \qquad I(0) = 0.1$$

$$R' = +I \qquad R(0) = 0$$



Vector Notation

$$u_1(t) = S(t)$$
$$u_2(t) = I(t)$$
$$u_3(t) = R(t)$$

$$\Longrightarrow \quad \begin{aligned} u_1' &= -a u_1 u_2 \\ u_2' &= a u_1 u_2 - b u_2 \\ u_3' &= b u_2 \end{aligned}$$

Setting $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $f(\bar{u}) = \begin{bmatrix} -a u_1 u_2 \\ a u_1 u_2 - b u_2 \\ b u_2 \end{bmatrix}$

$$\Longrightarrow \boxed{u' = f(u)}$$

Vector Notation of the system

System of First Order ODEs

System of 1st order ODEs:

$$\begin{aligned} x_1' &= f_1(t, x_1, \ldots, x_n) \\ &\vdots \\ x_n' &= f_n(t, x_1, \ldots, x_n) \end{aligned}$$

Vector notation:

$$\begin{aligned} \mathbf{x} &= [x_1, \ldots, x_n]^T \\ \mathbf{f} &= [f_1, \ldots, f_n]^T \\ \mathbf{x}' &= [x_1', \ldots, x_n']^T \end{aligned}$$

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \qquad (1)$$

n: dimension of system

$n = 2$: planar system

- (1) is autonomous if $\mathbf{f}$ does not depend on t
- (1) is non-autonomous if $\mathbf{f}$ depends on t

IVP: Existence and Uniqueness of Solution

Initial Value Problem:

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \qquad (2)$$

Thm.: If $\mathbf{f}$ is continuous in a region R and has continuous partial derivatives $\partial f_i / \partial x_j$ in R, (2) has a unique solution in R.

Example 1

Ex.:

$$\begin{aligned} x_1' &= -ax_1x_2 \\ x_2' &= ax_1x_2 - bx_2 \\ x_3' &= bx_2 \end{aligned}$$

$$\begin{aligned} \mathbf{x} &= [x_1, x_2, x_3]^T \\ \mathbf{f}(\mathbf{x}) &= [-ax_1x_2, ax_1x_2 - bx_2, bx_2]^T \end{aligned}$$

$\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is $3d$ autonomous system

Example 2

Ex.:

$$\begin{aligned} x' &= v \\ v' &= -x - 0.2v + 2\cos t \end{aligned}$$

is $2d$ non-autonomous system

Reduction of Higher Order Equations

Thm. Any system of higher order ODEs in explicit form can be transformed to a (larger) system of first order ODEs.

Example 3

Ex.: $$x''' + xx'' = \cos t \qquad (3)$$

Set $x_1 = x,\ x_2 = x',\ x_3 = x''$

$$\begin{aligned} \Rightarrow\ x_1' &= x' = x_2 \\ x_2' &= x'' = x_3 \\ x_3' &= x''' = -xx'' + \cos t \\ &= -x_1x_3 + \cos t \end{aligned}$$

Given a solution $x(t)$ of (3) $\Rightarrow$ $[x(t), x'(t), x''(t)]^T$ is solution of (4)

Hence equivalent system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -x_1x_3 + \cos t \end{aligned} \qquad (4)$$

Conversely: Given a solution $\mathbf{x}(t) = [x_1(t), x_2(t), x_3(t)]^T$ of (4) $\Rightarrow$ $x(t) = x_1(t)$ is a solution of (3)

General Higher Order ODEs

General Higher Order ODEs:

nth order ODE in explicit form:

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)})$$

Set $x_1 = x,\ x_2 = x',\ \ldots\ ,\ x_n = x^{(n-1)}$

$$\Rightarrow \begin{aligned} x_1' &= x' = x_2 \\ x_2' &= x'' = x_3 \\ &\vdots \\ x_{n-1}' &= x^{(n-1)} = x_n \\ x_n' &= x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}) \\ &= f(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

$\Rightarrow$ equivalent system:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= f(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

Exercise 8.1.1

Ex. 1: Is the system autonomous? What is the dimension?

$$\left.\begin{array}{rcl} x' & = & v \\ v' & = & -x - 0.02v + 2\cos t \end{array}\right\}$$ is non-autonomous ($\cos t$). Dimension: 2

Exercise 8.1.2

Ex. 2: Same questions as in Ex. 1

$$\left.\begin{array}{rcl}\theta' & = & \omega \\ \omega' & = & -(g/L)\sin\theta + (k/m)\omega\end{array}\right\}$$ is autonomous. Dimension: 2

Exercise 8.1.7

Ex. 7: Show that given functions are solutions of initial value problem

IVP: $\left\{ \begin{array}{lcl} x' & = & -4x+6y \\ y' & = & -3x+5y \end{array} \right\}$, $\left\{ \begin{array}{l} x(0)=0 \\ y(0)=1 \end{array} \right\}$; functions $\left\{ \begin{array}{lcl} x(t) & = & 2e^{2t}-2e^{-t} \\ y(t) & = & -e^{-t}+2e^{2t} \end{array} \right\}$

$$x'(t)=4e^{2t}+2e^{-t},\ -4x(t)+6y(t)=-4(2e^{2t}-2e^{-t})+6(-e^{-t}+2e^{2t})=4e^{2t}+2e^{-t}$$

$$y'(t)=e^{-t}+4e^{2t},\ -3x(t)+5y(t)=-3(2e^{2t}-2e^{-t})+5(-e^{-t}+2e^{2t})=e^{-t}+4e^{2t}$$

$$\text{IC: } x(0)=0,\ y(0)=1,$$

hence $x(t), y(t)$ are solutions of IVP