1. Find the solution of the following initial-value problem

   1.
$$\frac{dy}{dt} = ty^2, \quad y(0) = 1.$$

*Solution (S.O.V)*

$$\frac{dy}{dt} = ty^2 \quad \Rightarrow \quad (1/y^2)dy = tdt \quad \Rightarrow \quad \int (1/y^2)dy = \int tdt$$
$$\Rightarrow \quad -1/y = t^2/2 + c \quad \Rightarrow \quad y(t) = -\frac{2}{t^2+k}$$

$$y(0) = 1 = -\frac{2}{0+k} \quad \Rightarrow \quad k = -2,$$
$$\Rightarrow \quad y(t) = \frac{2}{2-t^2}$$

   2.
$$\frac{dy}{dt} = ry + a, \quad y(0) = y_0. \quad (r,\ a,\ y_0 \text{ parameters})$$

*Solution (S.O.V)*

$$\frac{dy}{dt} = ry + a \quad \Rightarrow \quad [1/(ry+a)]dy = dt \quad \Rightarrow \quad \int [1/(ry+a)]dy = \int dt$$
$$\Rightarrow \quad (\ln|ry+a|)/r = t + c \quad \Rightarrow \quad y(t) = ke^{rt} - a/r$$

$$y(0) = y_o = ke^0 - a/r \quad \Rightarrow \quad k = y_0 + a/r,$$
$$\Rightarrow \quad y(t) = (y_0 + a/r)e^{rt} - a/r$$

   3.
$$\frac{dy}{dt} = \frac{y}{t}, \quad y(1) = -2.$$

*Solution (S.O.V)*

$$\frac{dy}{dt} = \frac{y}{t} \quad \Rightarrow \quad (1/y)dy = (1/t)dt \quad \Rightarrow \quad \int (1/y)dy = \int (1/t)dt$$
$$\Rightarrow \quad \ln|y| = \ln|t| + c \quad \Rightarrow \quad y(t) = kt$$

$$y(1) = -2 = k * 1 \quad \Rightarrow \quad k = -2,$$
$$\Rightarrow \quad y(t) = -2t$$

4.
$$\frac{dy}{dt} = \frac{\sin t}{y}, \quad y(\frac{\pi}{2}) = 1.$$

*Solution (S.O.V)*

$$\frac{dy}{dt} = \frac{\sin t}{y} \quad \Rightarrow \quad ydy = \sin t dt \quad \Rightarrow \quad \int ydy = \int \sin t dt$$
$$\Rightarrow \quad y^2/2 = -\cos t + c \quad \Rightarrow \quad y(t) = \pm\sqrt{k - 2\ cost}$$

$$y(\pi/2) = 1 = \sqrt{k - 2\cos(\pi/2)} \quad \Rightarrow \quad k = 1,$$
$$\Rightarrow \quad y(t) = \sqrt{1 - 2\cos t}$$

5.
$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 1.$$

*Solution (S.O.V)*

$$\frac{dy}{dt} = 1 + y^2 \quad \Rightarrow \quad [1/(1+y^2)]dy = dt \quad \Rightarrow \quad \int [1/(1+y^2)]dy = \int dt$$
$$\Rightarrow \quad arctany = t + c \quad \Rightarrow \quad y(t) = \tan(t+c) + n\pi \ (n \text{ integer})$$

$$y(0) = 1 = \tan(0+c) + n\pi \quad \Rightarrow \quad c = \pi/4, n = 0,$$
$$\Rightarrow \quad y(t) = \tan(t + \pi/4)$$

6.
$$\frac{dy}{dt} = \frac{t}{y}, \quad y(0) = 1.$$

*Solution (S.O.V)*

$$\frac{dy}{dt} = \frac{t}{y} \quad \Rightarrow \quad ydy = tdt \quad \Rightarrow \quad \int ydy = \int tdt$$
$$\Rightarrow \quad y^2/2 = t^2/2 + c \quad \Rightarrow \quad y(t) = \pm\sqrt{t^2 + k}$$

$$y(0) = 1 = \sqrt{0^2 + k} \quad \Rightarrow \quad k = 1$$
$$\Rightarrow \quad y(t) = \sqrt{t^2 + 1}$$

7.
$$\frac{dy}{dt} + \cos t y = \frac{1}{2}\sin 2t, \quad y(0) = 1.$$

*Solution (Variation of Parameter)*

$$y_h(x) = \exp\left(\int (-\cos t)dt\right) = e^{-\sin t} \quad \Rightarrow \quad v(t) = \int \frac{1}{2}\sin 2te^{\sin t}dt = e^{\sin t}(\sin t - 1)$$
$$\Rightarrow \quad y_p(t) = \sin t - 1 \quad \Rightarrow \quad y(t) = ke^{-\sin t} + \sin t - 1$$

$$y(0) = 1 = k - 1 \quad \Rightarrow \quad k = 2$$
$$\Rightarrow \quad y(t) = 2e^{-\sin t} + \sin t - 1$$

8.
$$\frac{dy}{dt} + 2ty = 2t^3, \quad y(0) = -1.$$

*Solution (Variation of Parameter)*

$$y_h(x) = \exp\left(\int (-2t)dt\right) = e^{-t^2} \quad \Rightarrow \quad v(t) = \int 2t^3 e^{t^2} dt = e^{t^2}(t^2 - 1)$$
$$\Rightarrow \quad y_p(t) = t^2 - 1 \quad \Rightarrow \quad y(t) = ke^{-t^2} + t^2 - 1$$

$$y(0) = -1 = k - 1 \quad \Rightarrow \quad k = 0$$
$$\Rightarrow \quad y(t) = t^2 - 1$$

9.
$$\frac{dy}{dt} + \frac{y}{1+t} = 2, \quad y(0) = 1.$$

*Solution (Integrating Factor)*

$$u(t) = \exp\left(\int [1/(1+t)]dt\right) = 1 + t$$

$$\text{multiply ODE by } u \quad \Rightarrow \quad [(1+t)y]' = 2(1+t) \quad \Rightarrow \quad (1+t)y = \int 2(1+t)dt$$
$$\Rightarrow \quad y(t) = (c + 2t + t^2)/(1+t)$$

$$y(0) = 1 = c \quad \Rightarrow \quad c = 1$$
$$\Rightarrow \quad y(t) = (1 + 2t + t^2)/(1+t) \quad \Rightarrow \quad y(t) = 1 + t$$

10.
$$\frac{dy}{dt} - \frac{n}{t}y = e^t t^n, \quad y(1) = 1.$$

*Solution (Integrating Factor)*

$$u(t) = \exp\left(\int (-n/t)dt\right) = t^{-n}$$

$$\text{multiply ODE by } u \quad \Rightarrow \quad [t^{-n}y]' = e^t \quad \Rightarrow \quad t^{-n}y = \int e^t dt$$
$$\Rightarrow \quad y(t) = t^n(c + e^t)$$

$$y(1) = 1 = c + e \quad \Rightarrow \quad c = 1 - e$$
$$\Rightarrow \quad y(t) = t^n(1 - e + e^t)$$

2. A tank initially contains 50 gal of sugar water having a concentration of 2 lb of sugar for each gal of water. At time zero, pure water begins pouring into the tank at a rate of 2 gal per minute. Simultaneously, a drain is opened at the bottom of the tank so that the volume sugar-waater solution in the tank remains constant.

   (a) How much sugar is in tank after 10 minutes?

   (b) How long will it take the sugar content in the tank to dip below 20 lb?

   (c) What will be the eventual sugar content in the tank?

   **Note:** It was a homework problem. You should get $x(t) = 100e^{-\frac{t}{25}}$.

3. Suppose that $y$ is a solution to the initial value problem

$$y' = (y^2 - 1)e^{ty} \text{ and } y(1) = 0.$$

   Show that $-1 < y(t) < 1$ for all $t$ for which $y$ is defined.
   **Note:** We solved it in class.

4. Suppose a population is growing according to the logistic equation

$$\frac{dP}{dt} = f(P) \quad \text{where } f(P) = r_0\left(1 - \frac{P}{K}\right)P$$

   with $r_0$ being the natural reproductive rate and $K$ being the carrying capacity. Perform each of the following tasks without the aid of technology.

   (i) Sketch a graph of $f(P)$

   (ii) Use the graph of $f$ to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

   (iii) Sketch the equilibrium solutions in the $t$-$P$ plans. These equilibrium solutions divide the $t$-$P$ plane into regions. Sketch at least one solution trajectory in each of these regions.

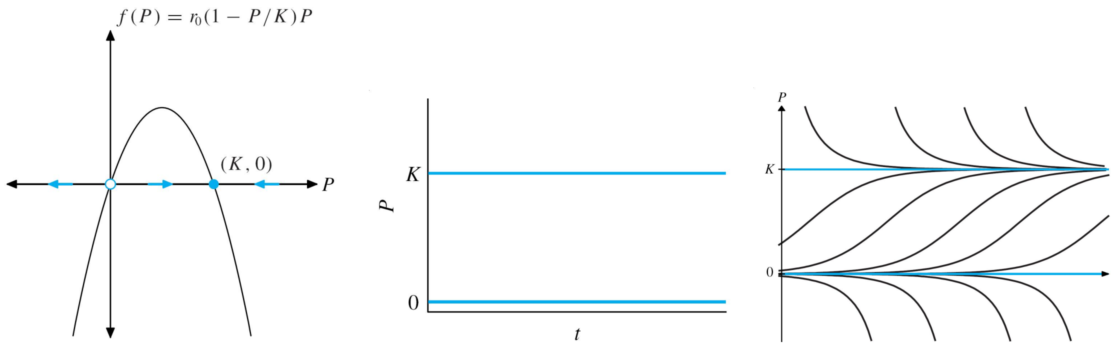
   *Solution* (i) The graph of $f(P)$ and (ii) the associated phase line are shown in Figure (left). (ii) The equilibrium points are where $f(P) = 0$, or at $P_1 = 0$ and $P_2 = K$. Note that

$$f(P) > 0 \text{ if } 0 < P < K \quad \text{and} \quad f(P) < 0 \text{ if } P < 0 \text{ or } P > K$$

   Hence $P_1 = 0$ is un unstable equilibrium point and $P_2 = K$ is stable. (iii) The equilibrium solutions are

$$P_1(t) = 0 \quad \text{and} \quad P_2(t) = K.$$

   The graph of the equilibrium solutions are shown in Figure (center). The solution curves are sketched in Figure (right), where any solution with a positive population must stay positive and must tend to $K$ as $t \mapsto \infty$.




5. Find the solution of the following initial-value problem

a. $y'' - 3y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1.$

*Solution* DE, its Characteristic Equation and roots

$$y'' - 3y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 - 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = 2, \lambda_2 = 1$$

The general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{2t} + C_2 e^t \quad \Rightarrow y'(t) = 2C_1 e^{2t} + C_2 e^t$$

ICs: $y(0) = 2 = C_1 + C_2$ and $y'(0) = 1 = 2C_1 + C_2$ imply

$$C_1 = -1, C_2 = 3 \quad \Rightarrow y(t) = -e^{2t} + 3e^t$$

b. $y'' + 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 3.$

*Solution* DE, its Characteristic Equation and roots

$$y'' + 2y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -1 \pm i$$

The general solution is

$$y(t) = e^{at}(C_1 \cos(bt) + C_2 \sin(bt)) = e^{-t}(C_1 \cos(t) + C_2 \sin(t))$$
$$\Rightarrow y'(t) = -e^{-t}(C_1 \cos(t) + C_2 \sin(t)) + e^{-t}(-C_1 \sin(t) + C_2 \cos(t))$$

ICs: $y(0) = 2 = C_1$ and $y'(0) = 3 = -C_1 + C_2$ imply

$$C_1 = 2, C_2 = 5 \quad \Rightarrow y(t) = e^{-t}(2\cos(t) + 5\sin(t))$$

c. $y'' - 2y' + y = 0, \quad y(0) = 2, \quad y'(0) = -1.$

*Solution* DE, its Characteristic Equation and roots

$$y'' - 2y' + y = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 1$$

The general solution is

$$y(t) = (C_1 + C_2 t)e^{\lambda_1 t} = (C_1 + C_2 t)e^t$$
$$\Rightarrow y'(t) = (C_1 + C_2 t)e^t + C_2 e^t$$

ICs: $y(0) = 2 = C_1$ and $y'(0) = -1 = C_1 + C_2$ imply

$$C_1 = 2, C_2 = -3 \quad \Rightarrow y(t) = (2 - 3t)e^t.$$

6. A 0.1 kg mass is attached to a spring having a spring constant 3.6 kg/s. The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present, find the amplitude, frequency, and phase of the resulting motion. Plot the solution.

*Solution* Subsitute $m$ =0.1 kg and $k$ =3.6 kg/s in $my'' + ky = 0$ to obtain $0.1y'' + 3.6y = 0$, or

$$y'' + 36y = 0$$

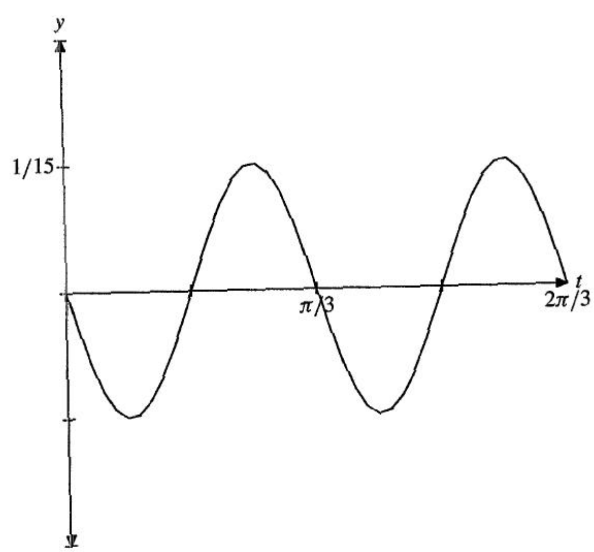
The general solution is

$$y(t) = C_1 \cos 6t + C_2 \sin 6t.$$

The initial displacement is zero, so $y(0) = 0 = C_1$. The initial velocity is $-0.4$ m/s, so $y'(0) = -0.4 = 6C_2$, leading to $C_2 = -1/15$. Thus the solution is

$$y(t) = -\frac{1}{15}\sin 6t = \frac{1}{15}\cos(6t + \pi/2)$$

which has amplitude 1/15, frequency 6 rad/s, and $-\pi/2$ phase.



7. Find the solution of the following initial-value problems

   a. $y'' + 3y' + 2y = 3e^{-4t}, \quad y(0) = 1, \quad y'(0) = 0.$

   *Solution* The homogeneous equation, its Characteristic Equation and roots

$$y'' + 3y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = -2, \lambda_2 = -1$$

   The homogeneous solution is

$$y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-2t} + C_2 e^{-t}$$

   The particular solution $y_p = Ae^{-4t}$ has derivatives $y_p' = -4Ae^{-4t}$ and $y_p'' = 16Ae^{-4t}$, which when subsitituted into the equation provides

$$y_p'' + 3y_p' + 2y_p = 3e^{-4t} \quad \Rightarrow \quad 16Ae^{-4t} + 3(-4Ae^{-4t}) + 2(Ae^{-4t}) = 3e^{-4t} \quad \Rightarrow \quad A = \frac{1}{2}$$

   Thus, a particular solution is $y_p = \frac{1}{2}e^{-4t}$. This leads to the general solution

$$y(t) = y_h(t) + y_p(t) = C_1 e^{-2t} + C_2 e^{-t} + \frac{1}{2}e^{-4t}$$

   ICs: $y(0) = 1 = C_1 + C_2 + \frac{1}{2}$ and $y'(0) = 0 = -2C_1 - C_2 - 2$ imply

$$C_1 = -\frac{5}{2}, C_2 = 3 \quad \Rightarrow y(t) = -\frac{5}{2}e^{-2t} + 3e^{-t} + \frac{1}{2}e^{-4t}$$

   b. $y'' + 2y' + 2y = 2\cos 2t, \quad y(0) = -2, \quad y'(0) = 0.$

   *Solution* The homogeneous equation, its Characteristic Equation and roots

$$y'' + 2y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = -1 + i, \lambda_2 = -1 - i$$

   The homogeneous solution is

$$y_h(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) = e^{-t}(C_1 \cos t + C_2 \sin t)$$

   The particular solution $y_p = A_1 \cos 2t + A_2 \sin 2t$ has derivatives

$$y_p' = -2A_1 \sin 2t + 2A_2 \cos 2t, \quad y_p'' = -4A_1 \cos 2t - 4A_2 \sin 2t,$$

   which when subsitituted into the equation provides

$$\begin{aligned} y_p'' + 2y_p' + 2y_p = 2\cos 2t \quad &\Rightarrow \quad (-4A_1 \cos 2t - 4A_2 \sin 2t) \\ &+ 2(-2A_1 \sin 2t + 2A_2 \cos 2t) + 2(A_1 \cos 2t + A_2 \sin 2t) = 2\cos 2t \\ \Rightarrow \quad -A_1 + 2A_2 = 1, \quad 2A_1 + A_2 = 0 \quad &\Rightarrow \quad A_1 = -\frac{1}{5}, \quad A_2 = \frac{2}{5} \end{aligned}$$

Thus, a particular solution is $y_p = -\frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$. This leads to the general solution

$$y(t) = y_h(t) + y_p(t) = e^{-t}(C_1 \cos t + C_2 \sin t) - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$$

ICs: $y(0) = -2 = C_1 - \frac{1}{5}$ and $y'(0) = 0 = C_2 - C_1 + \frac{4}{5}$ imply

$$C_1 = -\frac{9}{5}, C_2 = -\frac{13}{5} \quad \Rightarrow y(t) = e^{-t}\left(-\frac{9}{5}\cos t - \frac{13}{5}\sin t\right) - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$$

c. $y'' - 2y' + y = t^3, \quad y(0) = 1, \quad y'(0) = 0.$

*Solution* The homogeneous equation, its Characteristic Equation and roots

$$y'' - 2y' + y = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 1$$

The homogeneous solution is

$$y_h(t) = (C_1 + C_2 t)e^{\lambda t} = (C_1 + C_2 t)e^t$$

The particular solution $y_p = at^3 + bt^2 + ct + d$ has derivatives

$$y_p' = 3at^2 + 2bt + c, \quad y_p'' = 6at + 2b,$$

which when subsitituted into the equation provides

$$\begin{aligned} &y_p'' - 2y_p' + y_p = t^3 \quad \Rightarrow \quad (6at + 2b) - 2(3at^2 + 2bt + c) + (at^3 + bt^2 + ct + d) = t^3 \\ &\quad \Rightarrow \quad a = 1, \quad -6a + b = 0, \quad 6a - 4b + c = 0, \quad 2b - 2c + d = 0 \\ &\quad \Rightarrow \quad a = 1, b = 6, c = 18, d = 24 \end{aligned}$$

Thus, a particular solution is $y_p = t^3 + 6t^2 + 18t + 24$. This leads to the general solution

$$y(t) = y_h(t) + y_p(t) = (C_1 + C_2 t)e^t + t^3 + 6t^2 + 18t + 24$$

ICs: $y(0) = 1 = C_1 + 24$ and $y'(0) = 0 = C_2 + C_1 + 18$ imply

$$C_1 = -23, C_2 = 5 \quad \Rightarrow \quad y(t) = (-23 + 5t)e^t + t^3 + 6t^2 + 18t + 24$$

d. $y'' + 4y' + 4y = 2e^{-2t}, \quad y(0) = 0, \quad y'(0) = 1.$

*Solution* The homogeneous equation, its Characteristic Equation and roots

$$y'' + 4y' + 4y = 0 \quad \Rightarrow \quad \lambda^2 + 4\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = -2$$

The homogeneous solution is

$$y_h(t) = (C_1 + C_2 t)e^{\lambda t} = (C_1 + C_2 t)e^{-2t}$$

The forcing term $f(t) = 2e^{-2t}$ and $Ate^{-2t}$ are solutions of the homogeneous equations, so multiply by another factor of $t$ and try the particular solution $y_p = At^2e^{-2t}$. The derivatives of $y_p$ are

$$y_p' = 2Ae^{-2t}(t - t^2), \quad y_p'' = 2Ae^{-2t}(2t^2 - 4t + 1),$$

which when subsitituted into the equation provides

$$\begin{aligned} y_p'' + 4y_p' + 4y_p = 2e^{-2t} \quad &\Rightarrow \quad 2Ae^{-2t}(2t^2 - 4t + 1) + 4(2Ae^{-2t}(t - t^2)) + 4(At^2e^{-2t}) = 2e^{-2t} \\ \Rightarrow \quad &[2(2t^2 - 4t + 1) + 8(t - t^2) + 4t^2]A = 2 \\ \Rightarrow \quad &A = 1 \end{aligned}$$

Thus, a particular solution is $y_p = t^2e^{-2t}$. This leads to the general solution

$$y(t) = y_h(t) + y_p(t) = (C_1 + C_2t)e^{-2t} + t^2e^{-2t}$$

ICs: $y(0) = 0 = C_1$ and $y'(0) = 1 = C_2 - 2C_1$ imply

$$C_1 = 0, C_2 = 1 \quad \Rightarrow \quad y(t) = te^{-2t} + t^2e^{-2t}$$

8. Verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2y'' + 3ty' - 3y = 0, \quad \text{for } t > 0.$$

Use the variation of parameters to find the general solution to

$$t^2y'' + 3ty' - 3y = \frac{1}{t}, \quad \text{for } t > 0.$$

*Solution* We start by rewriting the equation

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = \frac{1}{t^3}$$

First, check $y_1(t) = t$ is a solution

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = (0) + \frac{3}{t}(1) - \frac{3}{t^2}(t) = 0$$

Check $y_2(t) = t^{-3}$ is a solution

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = (12t^{-5}) + \frac{3}{t}(-3t^{-4}) - \frac{3}{t^2}(t^{-3}) = 0$$

Calculate the Wronskian

$$W(t, t^{-3}) = \begin{vmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{vmatrix} = -4t^{-3}$$

Next

$$v_1' = \frac{-y_2 g(t)}{W} = \frac{-t^{-3}t^{-3}}{-4t^{-3}} = \frac{1}{4}t^{-3}$$
$$v_2' = \frac{y_1 g(t)}{W} = \frac{tt^{-3}}{-4t^{-3}} = -\frac{1}{4}t$$

Thus

$$v_1 = -\frac{1}{8}t^{-2}, \quad v_1 = -\frac{1}{8}t^2$$

Form

$$y_p = v_1 y_1 + v_2 y_2 = \left(-\frac{1}{8}t^{-2}\right) t + \left(-\frac{1}{8}t^2\right) t^{-3} = -\frac{1}{4t}$$

Thus, the general solution is

$$y(t) = y_h(t) + y_p(t) = C_1 t + \frac{C_2}{t^3} - \frac{1}{4t}$$

9. An undamped spring-mass system with external driving force is modeled with

$$x'' + 25x = 4\cos 5t.$$

The parameters of this equation are "tuned" so that the frequency of the driving force equals the natural frequency of the undriven system. Suppose that the mass is displaced one positive unit and released from rest.

(a) Find the position of the mass as a function of time. What part of the solution guarantees that this solution resonates?

(b) Sketch the solution found in part (a).

*Solution* (a) As in the notes, the particular solution is

$$x_p(t) = \frac{A}{2\omega_0} t \sin \omega_0 t = \frac{2}{5} t \sin 5t$$

The general solution of the homogeneous equation is

$$x_h(t) = C_1 \cos 5t + C_2 \sin 5t$$

So the solution has the form

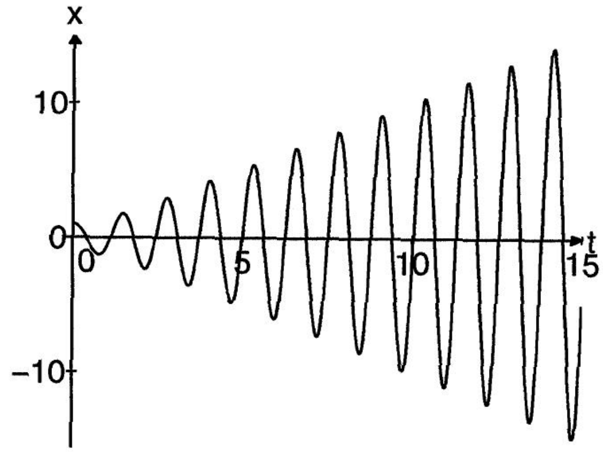
$$x(t) = x_h(t) + x_p(t) = C_1 \cos 5t + C_2 \sin 5t + \frac{2}{5} t \sin 5t$$

Apply I.C.s yields $1 = x(0) = C_1$ and $0 = x'(0) = 5C_2$. So

$$x(t) = \cos 5t + \frac{2}{5} t \sin 5t$$

The particular solution $x_p(t)$ has a factor of $t$ so its amplitude will grow, indicating a resonant solution.

(b)


x
10
0
−10
0
5
10
15
t