2. (b) \( Q = 0 \), \( u_x(0) = 0 \), \( u(L) = T \):
- \( u_{xx} = 0 \) \( \Rightarrow \) \( u(x) = C_1 x + C_2 \)
- \( u_x(0) = C_1 = 0 \) \( \Rightarrow \) \( u(x) = C_2 \)
- \( u(L) = T = C_2 \) \( \Rightarrow \) \( u(x) = T \)

(a), (d), (f), (h) \( \Rightarrow \) textbook
(e), (g) \( \Rightarrow \) solved in class

3. \( E(t) = \frac{C_0}{L} \int_0^L u(x,t) \, dx \)

(Solved in class)

4. (a) \( u_{xx} + 1 = 0 \) \( \Rightarrow \) \( u_{xx} = -1 \) \( \Rightarrow \) \( u_x = -x + C_1 \) \( \Rightarrow \) \( u(x) = -\frac{x^2}{2} + C_1 x + C_2 \)
- \( u(x,0,t) = 1 \) \( \Rightarrow \) \( u_x(0,t) = 0 + C_1 = 1 \) \( \Rightarrow \) \( C_1 = 1 \)
- \( u_x(L,t) = \beta \) \( \Rightarrow \) \( u_x(L,t) = L - 1 = \beta \) \( \Rightarrow \) \( \beta \) must be equal to \( 1 - L \). \( \beta = 1 - L \)

(b) \( u_{xx} = 0 \) \( \Rightarrow \) \( u_x = C_1 \) \( \Rightarrow \) \( u = C_1 x + C_2 \)
- \( u(x,0,t) = 1 \) \( \Rightarrow \) \( C_1 = 1 \) \( \Rightarrow \) \( u = C_1 x + C_2 = x + C_2 \)
- \( u_x(L,t) = \beta \) \( \Rightarrow \) \( \beta \) must be equal to \( 1 \) \( \Rightarrow \) \( \beta = 1 \)

Solution: \( u(x,t) = x + C_2 \)

No sources, flux of heat left is equal to flux on the right \( \Rightarrow \) energy conserved. Determine \( C_2 \) from initial condition.

Energy of initial data = energy of equilibrium

\[ E(0) = C_0 \int_0^L f(x) \, dx = C_0 \int_0^L u^0(x) \, dx = C_0 \int_0^L (x+C_2) \, dx = \frac{x^2}{2} + C_1 x \bigg|_0^L = \frac{L^2}{2} + C_1 L \Rightarrow \]

\[ C_2 = \frac{L}{L} \left[ \int_0^L f(x) \, dx - \frac{L^2}{2} \right] \Rightarrow \]

\[ u(x) = x + \frac{1}{L} \left[ \int_0^L f(x) \, dx - \frac{L^2}{2} \right] \]
(c) \( u_{xx} + x - \beta = 0 \Rightarrow u_{xx} = \beta - x \Rightarrow u_x = \beta x - \frac{x^2}{2} + c_1 \Rightarrow \)

\[
\begin{align*}
\text{u(x)} &= \beta \frac{x^2}{2} - \frac{x^3}{6} + c_1 x + c_2 \\
\text{u_x(0,t)} &= 0 \Rightarrow u_x(0) = c_1 = 0 \Rightarrow c_1 = 0 \\
\text{u_x(L)} &= 0 \Rightarrow u_x(L) = \beta L - \frac{L^2}{2} + 0 = 0 \Rightarrow \beta = \frac{L}{2}
\end{align*}
\]

Equilibrium solution exists if \( \beta = \frac{L}{2} \). In that case the equilibrium solution is given by

\[
\text{u(x)} = \frac{L}{2} \frac{x^2}{2} - \frac{x^3}{6} + c_2
\]

Is it possible to determine \( c_2 \)?

Is the energy conserved in this problem?

Answer: calculate \( \frac{dE}{dt} \) and check if it is 0.

\[
\frac{dE}{dt} = c_0 \int_0^L u_t A \, dx = c_0 A \int_0^L (u_{xx} + x - \frac{L}{2}) \, dx =
\]

\[
= c_0 A \left[ \frac{L}{2} \frac{x^2}{2} - \frac{x^3}{6} + \frac{L^2}{2} \right]_0^L = c_0 A \left[ \frac{L}{2} \frac{L^2}{2} - \frac{L^2}{2} \right] = 0
\]

(Insulated ends)

Since energy is conserved, \( E(0) = E(t) \), for all \( t \)

Thus, the total energy of \( u^e \) is equal to the energy of initial data \( u(x,0) = f(x) \). From here we determine \( c_2 \):

\[
\frac{d}{dt} \int_0^L u^e(x) A \, dx = c_0 A \int_0^L \left( \frac{L}{4} x^2 - \frac{x^3}{6} + C_2 \right) \, dx = E(0) = c_0 A \int_0^L f(x) \, dx
\]

\[
\Rightarrow \int_0^L \left( \frac{L}{4} x^2 - \frac{x^3}{6} + C_2 \right) \, dx = \int_0^L f(x) \, dx \Rightarrow \]

\[
\frac{L^3}{12} - \frac{L^4}{24} + C_2 L = \int_0^L f(x) \, dx \Rightarrow C_2 = \frac{L}{L} \left( \int_0^L f(x) \, dx - \frac{L^4}{24} \right)
\]

\[
\text{u^e(x)} = \frac{L}{4} x^2 - \frac{x^3}{6} + \frac{L}{L} \left( \int_0^L f(x) \, dx - \frac{L^4}{24} \right)
\]
1. $L$, $L_2$ linear, $L(u) = a_1 L_1(u) + a_2 L_2(u)$ (lin. comb of lin. operators)
   
   Show that $L$ is linear. More precisely, show that
   
   $L\left( a_1 u_1 + a_2 u_2 \right) = a_1 L(u_1) + a_2 L(u_2)$.
   
   **Proof:**
   
   $L\left( a_1 u_1 + a_2 u_2 \right)$ by definition
   
   $= a_1 L_1(u_1) + a_2 L_2(u_2)$
   
   $(L_1, L_2$ linear $)
   
   = a_1 \left[ L_1(u_1) + a_2 \frac{d}{dx} L_2(u_2) \right] + a_2 \left[ L_1(u_2) + a_2 \frac{d}{dx} L_2(u_2) \right]
   
   regroup terms
   
   $= a_1 \left[ L_1(u_1) + a_2 \frac{d}{dx} L_2(u_2) \right] + a_2 \left[ L_1(u_2) + a_2 \frac{d}{dx} L_2(u_2) \right]
   
   by definition of $L$
   
   $= a_1 L(u_1) + a_2 L(u_2)$

2. Show that $L(a_1 u_1 + a_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$ for $L(u) = \frac{\partial}{\partial x} (\psi(x) \frac{\partial u}{\partial x})$
   
   or: show that
   
   $\frac{\partial}{\partial x} (\psi(x) \frac{\partial u}{\partial x}) = c_1 \frac{\partial}{\partial x} (\psi(x) \frac{\partial u}{\partial x}) + c_2 \frac{\partial}{\partial x} (\psi(x) \frac{\partial u}{\partial x})$.
   
   Go step by step to show this.

3. (a) Suppose $Q = \alpha(x,t) u + \beta(x,t)$. Define $L(u) = \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} + Q$.
   
   Show that $L(a_1 u_1 + a_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$.
   
   (b) Show that $L(0) = 0$ if $\beta = 0$.

4. In class.