Homework 10: March 22, 2017

Wave Equation

1. Prove that $u(x,t) = F(x-ct) + G(x+ct)$ solves the wave equation $u_{tt} = c^2 u_{xx}$, where F and G are arbitrary functions. What are $u(x,0)$ and $u_t(x,0)$ for $u(x,t) = F(x-ct) + G(x+ct)$?

2. Solve $u_{tt} = 9u_{xx}$ for $x \in R$, $t > 0$, with the initial data $u(x,0) = 0$ and $u_t(x,0) = x$. Sketch and specify the **Domain of dependence** of point $(x,t) = (2,3)$ and the **Range of influence** of point $(x_0, 0) = (0,0)$.

3. Suppose that an infinite vibrating strung is initially stretched into the shape of a single rectangular pulse and is let go from rest. The corresponding initial conditions are

$$u(x,0) = f(x) = \begin{cases} 1, & |x| < h, \\ 0, & |x| > h, \end{cases} \qquad and \quad u_t(x,0) = g(x) = 0.$$

The solution consists of the initial pulse being separated into two pulses with half the amplitude, moving in opposite directions. At which time will the two pulses completely separate for the first time?

4. The sound speed in air is 343 meters per second. How long will it take an observer to hear an explosion that occurred 1 mile (i.e., 1609.34 meters) away?

5. The sound speed in water is 1,484 meters per second. How long will it take a submarine to feel an explosion that took place 1 mile away?

6. The total energy $E(t)$ of a string is defined to be the sum of the kinetic energy $E_K(t)$ and the potential energy $E_P(t)$, where

$$E_K(t) = \frac{1}{2}\int_0^L (u_t)^2 dx, \quad E_P(t) = \frac{1}{2}c^2 \int_0^L (u_x)^2 dx.$$

Show that the total energy of the vibrating string with fixed ends is conserved.

Hint: Start with the wave equation $u_{tt} = c^2 u_{xx}$, multiply the equation by u_t and integrate with respect to x from 0 to L. Use integration by parts on the term with $u_{xx}u_t$. Ask in class if you do not know how to solve this.