Note: There will likely be only 5 and not 6 problems on the actual exam.

Practice Midterm Exam 1

1. (10) Consider the problem:

$$u_t = u_{xx} + x - \alpha, \quad u(x,0) = f(x),\ u_x(0,t) = 0,\ u_x(L,t) = 0.$$

For which value of α is there an equilibrium solution? You do not have to find the equilibrium solution.

2. (10) Is the wave equation operator $L(u)$ below linear? Prove your statement.

$$L(u) = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}.$$

3. (15 pts) Solve the following eigenvalue problem:

$$\begin{aligned} \frac{d^2\phi}{dx^2} &= -\lambda\phi, \quad x \in (-L, L) \\ \phi(-L) &= \phi(L) \\ \frac{d\phi}{dx}(-L) &= \frac{d\phi}{dx}(L). \end{aligned}$$

4. Consider the heat flow in a wire of length L, thermal diffusivity 5, modeled by the heat equation $u_t = 5u_{xx}$ defined for $x \in (0, L)$ and $t > 0$, subject to the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$.

(a) (15 pts) Solve the initial boundary value problem if the temperature is initially

$$u(x,0) = \sin(\frac{4\pi x}{L})\ \cos(\frac{\pi x}{L}).$$

Hint: Use the following trigonometric identity: $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$ to express the initial data as the sum of two sine functions.

(b) (5 pts) What is the limit as $t \to \infty$ of the temperature distribution in the wire?

(c) (5 pts) Will the temperature distribution in the wire ever exceed 1 degree? Explain.

5. (15 pts) Suppose $u_t = 10u_{xx} + 1$, $u(x,0) = x^2$, $u_x(0,t) = 4$, $u_x(L,t) = 6$, where u denotes the temperature in a rod of length $L = 1$, with specific heat $c = 1$ and density $\rho = 1$, and cross-sectional area $A = 2$. Calculate the total thermal energy in the one-dimensional rod (as a function of time).

6. (a) (15 pts) Find the temperature distribution in a wire of length L, with thermal diffusivity $k = 1$ and insulated end-points, and with the initial temperature distribution given by

$$u(x,0) = 3 + \cos\frac{2\pi x}{L}.$$

(b) (10 pts) What is the limit as $t \to \infty$ of the temperature distribution in the wire?

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①

① $\frac{dE(t)}{dt} = c\rho \int_0^L \frac{\partial u}{\partial t} A\,dx = c\rho \int_0^L [u_{xx} + (x-\alpha)] A\,dx =$

$= c\rho A \left[\underbrace{[u_x]_0^L}_{=0} + \left(\frac{x^2}{2} - \alpha x \right) \right]_0^L = 0$

$\Rightarrow \boxed{\frac{L}{2} = \alpha}$

② $L(c_1 u_1 + c_2 u_2) \stackrel{?}{=} c_1 L(u_1) + c_2 L(u_2)$ .... (see next page)

③ $\phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$\phi(-L) = \cancel{A\cos\sqrt{\lambda}L} - B\sin\sqrt{\lambda}L = \cancel{A\cos\sqrt{\lambda}L} + B\sin\sqrt{\lambda}L$

(*) $\Rightarrow 2B\sin\sqrt{\lambda}L = 0 \rightarrow B = 0$

$\rightarrow \sqrt{\lambda}L = n\pi,\ n = 1, 2, \dots$

$\phi' = -\sqrt{\lambda}A\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$

$\phi'(-L) = +\sqrt{\lambda}A\sin\sqrt{\lambda}L + \cancel{B\sqrt{\lambda}\cos\sqrt{\lambda}L} =$

$\phi'(L) = -\sqrt{\lambda}A\sin\sqrt{\lambda}L + \cancel{B\sqrt{\lambda}\cos\sqrt{\lambda}L}$) =

(**) $\Rightarrow 2\sqrt{\lambda}A\sin\sqrt{\lambda}L = 0 \rightarrow A = 0$

$\rightarrow \sqrt{\lambda}L = n\pi,\ n = 1, 2, \dots$

~~B=0~~

To have both conditions satisfied at the same time

$\boxed{\sqrt{\lambda}L = n\pi,\quad n = 1, 2, \dots}$ $\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}$

$\phi_n(x) = A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}$

$\phi_n(x) = \cos\frac{n\pi x}{L},\ \sin\frac{n\pi x}{L}$, $n = 1, 2, 3, \dots$

For $\lambda = 0 \Rightarrow \phi_0 = \cos 0 = 1$

① ②

④ (a) $u(x,t) = \frac{1}{2}\sin\frac{5\pi x}{L} e^{-5\left(\frac{5\pi}{L}\right)^2 t} + \frac{1}{2}\sin\frac{3\pi x}{L} e^{-5\left(\frac{3\pi}{L}\right)^2 t}$

(b) $\lim_{t\to\infty} u(x,t) = 0$

(c) Since $e^{-\xi} \le 1$ for $\xi \ge 0$, and since $|\sin\alpha| \le 1$ we get that $|u(x,t)| \le 1$, $\forall (x,t)$, $t>0$

This means that there is no heat source in this problem that would ~~keep~~ make the max of initial temperature increase in time.

⑤ $\frac{dE(t)}{dt} = c\rho A \int_0^L (10u_{xx}+1)\,dx = 2\left[[u_x]_{x=0}^{L} + [x]_{x=0}^{L} \right]$

$= 2\left[(6-4)+L\right] = 2\left[L+2\right]$

$E(t) = E(0) + (4+2L)t = 2\int_0^L x^2\,dx + (4+2L)t = \frac{L^3}{3} + (4+2L)t$

⑥ (a) $u(x,t) = 3 + \cos\frac{2\pi x}{L} e^{-\left(\frac{2\pi}{L}\right)^2 t}$

(b) $\lim_{t\to\infty} u(x,t) = 3$

$\frac{\partial^2(f_1+f_2)}{\partial t^2} = \frac{\partial^2 f_1}{\partial t^2} + \frac{\partial^2 f_2}{\partial t^2}$

② $L(c_1u_1+c_2u_2) = \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)(c_1u_1+c_2u_2) = \frac{\partial^2(c_1u_1)}{\partial t^2} - c^2\frac{\partial^2(c_1u_1)}{\partial x^2} +$

$+ \frac{\partial^2(c_2u_2)}{\partial t^2} - c^2\frac{\partial^2(c_2u_2)}{\partial x^2} = c_1\frac{\partial^2 u_1}{\partial t^2} + c_1c^2\frac{\partial^2 u_1}{\partial x^2} + c_2\frac{\partial^2 u_2}{\partial t^2} - c_2c^2\frac{\partial^2 u_2}{\partial x^2} =$

$\frac{\partial^2(c_1u_1)}{\partial t^2} = c_1\frac{\partial^2 u_1}{\partial t^2}$

$= c_1\left(\frac{\partial^2 u_1}{\partial t^2} - c^2\frac{\partial^2 u_1}{\partial x^2}\right) + c_2\left(\frac{\partial^2 u_2}{\partial t^2} - c^2\frac{\partial^2 u_2}{\partial x^2}\right) = c_1L(u_1) + c_2L(u_2)$