NAME (please print): ______________________________

Quiz 2 (Sample)

1. (30 points) Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions: $Q = x^2$, $u(0) = T$, $u_x(L) = 0$.

2. Consider the problem:

$$u_t = u_{xx} + x - \beta, \quad u(x,0) = f(x),\ u_x(0,t) = 0,\ u_x(L,t) = 0.$$

(a) (30 points) For what value of β is there an equilibrium solution?

(b) (30 points) For the value of β found in (a), calculate the equilibrium solution.

3. Suppose $u_t = 3u_{xx} + x$, $u(x,0) = x$, $u_x(0,t) = 1$, $u_x(L,t) = 2$, where u denotes the temperature in a rod of length $L = 1$, with specific hear $c = 1$ and density $\rho = 1$, and cross-sectional area $A = 2$. Calculate the total thermal energy in the one-dimensional rod (as a function of time).

4. Is the differential operator $L(u) = u_t - 2xu_{xx}$ linear? Prove it.

NAME (please print): Answer Key

Quiz 2 (Sample)

1. (30 points) Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions: $Q = x^2$, $u(0) = T$, $u_x(L) = 0$.

$c\rho u_t = K_0 u_{xx} + x^2 \Rightarrow$ EQUILIBRIUM SOL'N SATISFIES: $K_0 u_{xx} + x^2 = 0$

$\Rightarrow u_{xx} = -\frac{x^2}{K_0} \Rightarrow u_x(x) = -\frac{1}{K_0}\frac{x^3}{3} + C_1 \Rightarrow \boxed{u^e(x) = -\frac{1}{K_0}\frac{x^4}{12} + C_1 x + C_2}$

$u(0) = T \Rightarrow u^e(0) = C_2 = T \Rightarrow \boxed{C_2 = T}$

$u_x(L) = 0 \Rightarrow -\frac{1}{K_0}\frac{L^3}{3} + C_1 = 0 \Rightarrow \boxed{C_1 = \frac{1}{K_0}\frac{L^3}{3}} \Rightarrow \boxed{u^e(x) = -\frac{1}{K_0}\frac{x^4}{12} + \frac{1}{K_0}\frac{L^3}{3}x + T}$

2. Consider the problem:

$$u_t = u_{xx} + x - \beta, \quad u(x,0) = f(x),\ u_x(0,t) = 0,\ u_x(L,t) = 0.$$

(a) (30 points) For what value of β is there an equilibrium solution?

(b) (30 points) For the value of β found in (a), calculate the equilibrium solution.

(a) Equilibrium: $u_{xx} + x - \beta = 0 \Rightarrow u_{xx} = -x + \beta \Rightarrow u_x = -\frac{x^2}{2} + \beta x + c_1 \Rightarrow$

$\Rightarrow \boxed{u(x) = -\frac{x^3}{6} + \beta\frac{x^2}{2} + c_1 x + c_2}$ $u_x(0) = 0 \Rightarrow c_1 = 0$, $u_x(L) = 0 \Rightarrow \boxed{\beta = \frac{L}{2}}$

$\Rightarrow \boxed{u(x) = -\frac{x^3}{6} + \frac{L}{2}\frac{x^2}{2} + c_2}$ EQUILIBRIUM EXISTS IF $\beta = \frac{L}{2}$.

(b) E is conserved (show this!) $\Rightarrow E(t) = E(0) \Rightarrow \int_0^L \left(-\frac{x^3}{6} + \frac{L}{2}\frac{x^2}{2} + c_2\right)dx = \int_0^L f(x)\,dx$

$\boxed{c_2 = \frac{1}{L}\int_0^L f\,dx - \frac{L^3}{24}}$

3. Suppose $u_t = 3u_{xx} + x$, $u(x,0) = x$, $u_x(0,t) = 1$, $u_x(L,t) = 2$, where u denotes the temperature in a rod of length $L = 1$, with specific hear $c = 1$ and density $\rho = 1$, and cross-sectional area $A = 2$. Calculate the total thermal energy in the one-dimensional rod (as a function of time).

$E(t) = 1\cdot 1\int_0^1 u(x,t)\cdot 2\,dx \Rightarrow \frac{dE(t)}{dt} = 2\int_0^1 u_t\,dx = 2\int_0^1 (3u_{xx} + x)\,dx =$

$= 6\left[u_x\right]_0^1 + 2\cdot\left[\frac{x^2}{2}\right]_0^1 = 6[2-1] + 1 = 7$

$\frac{dE(t)}{dt} = 7 \Rightarrow E(t) = E(0) + 7t = \int_0^1 Af(x)\,dx + 7t = 2\int_0^1 x\,dx + 7t = 1 + 7t$

$\Rightarrow \boxed{E(t) = 1 + 7t}$

4. Is the differential operator $L(u) = u_t - 2xu_{xx}$ linear? Prove it.

Yes. $L(u) = u_t - 2x u_{xx}$ is linear. Proof: Show that

$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)$.

Indeed: $L(c_1u_1 + c_2u_2) = \frac{\partial}{\partial t}(c_1u_1 + c_2u_2) - 2x\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) =$

$= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - 2x\cdot c_1\frac{\partial^2 u_1}{\partial x^2} - 2x\,c_2\cdot\frac{\partial^2 u_2}{\partial x^2} =$

$= c_1\left\{\frac{\partial u_1}{\partial t} - 2x\frac{\partial^2 u_1}{\partial x^2}\right\} + c_2\left\{\frac{\partial u_2}{\partial t} - 2x\frac{\partial^2 u_2}{\partial x^2}\right\} = c_1L(u_1) + c_2L(u_2)$ //