NAME (please print): ____________________

Quiz 2

1. Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties $c = 1, \rho = 1, K_0 = 1$, and with the following sources and boundary conditions: $Q = x$, $u(0) = 4$, $u_x(L) = 0$.

2. If both ends of the rod are insulated, derive using the heat equation, that the total thermal energy in the rod is constant.

3. Suppose $u_t = 10u_{xx} + 1$, $u(x, 0) = x^2$, $u_x(0, t) = 4$, $u_x(L, t) = 6$, where u denotes the temperature in a rod of length $L = 1$, with specific heat $c = 1$ and density $\rho = 1$, and cross-sectional area $A = 2$. Calculate the total thermal energy in the one-dimensional rod (as a function of time).

4. Let L be a linear operator, and c_1 and c_2 arbitrary constants. Prove that if u_1 and u_2 are two solutions of $L(u) = 0$, then $c_1u_1 + c_2u_2$ is again a solution of $L(u) = 0$. Show all the steps in the proof.

QUIZ 2 (Answer Key)

(1) $c=1, \rho=1, K_0=1, Q=x, u(0)=4, u_x(L)=0.$

$u_t = u_{xx} + x$

Equilibrium solution satisfies: $\begin{cases} u_{xx}+x=0 \\ u(0)=4 \\ u_x(L)=0 \end{cases}$

$\Rightarrow u_{xx} = -x \Rightarrow u_x = -\frac{x^2}{2} + C_1 \Rightarrow u = -\frac{x^3}{6} + C_1 x + C_2$

$u(0)=4 \Rightarrow u(0) = C_2 = 4 \Rightarrow \boxed{C_2 = 4}$

$u_x(L)=0 \Rightarrow -\frac{L^2}{2} + C_1 = 0 \Rightarrow \boxed{C_1 = \frac{L^2}{2}} \Rightarrow \boxed{u^e(x) = -\frac{x^3}{6} + \frac{L^2}{2}x + 4}$

(2) $\rho c u_t = K_0 u_{xx}$, $u_x(0,t)=0$, $u_x(L,t)=0$ (insulated ends)

$E(t) = c\rho \int_0^L u(x,t) A\,dx$

$\frac{dE(t)}{dt} = c\rho \int_0^L u_t(x,t) A\,dx = \int_0^L K_0 u_{xx}\,dx = K_0 \left[u_x\right]_{x=0}^{x=L} =$

$= K_0 \left[u_x(0,t) - u_x(L,t)\right] = K_0 [0-0] = 0$

$\Rightarrow E(t)$ is constant in time.

(3) $u_t = 10u_{xx} + 1$, $u(x,0)=x^2$, $u_x(0,t)=4$, $u_x(L,t)=6$, $L=1, c=1, \rho=1, A=2$

$\frac{dE(t)}{dt} = 1\cdot 1 \int_0^1 [10u_{xx}+1]\cdot 2\,dx = 2\left[10\left[u_x\right]_{x=0}^{x=1} + \left[x\right]_{x=0}^{x=1}\right] =$

$= 2\left[10\cdot(6-4)+1\right] = 2\cdot[21] = 42$

$\underline{\underline{E(t)}} = E(0) + 42t = \underbrace{\int_0^1 x^2 \cdot 2\,dx}_{E(0)} + 42t = 2\cdot\left[\frac{x^3}{3}\right]_0^1 + 42t = \underline{\underline{\frac{2}{3} + 42t}}$

(4) $L(c_1u_1 + c_2u_2) \underset{\uparrow}{=} c_1L(u_1) + c_2L(u_2) \underset{\uparrow}{=} c_1\cdot 0 + c_2\cdot 0 = 0$

L Linear

u_1 & u_2 solutions of $L(u)=0$

$\Rightarrow c_1u_1 + c_2u_2$ is a solution of $\underline{\underline{L(u)=0}}$